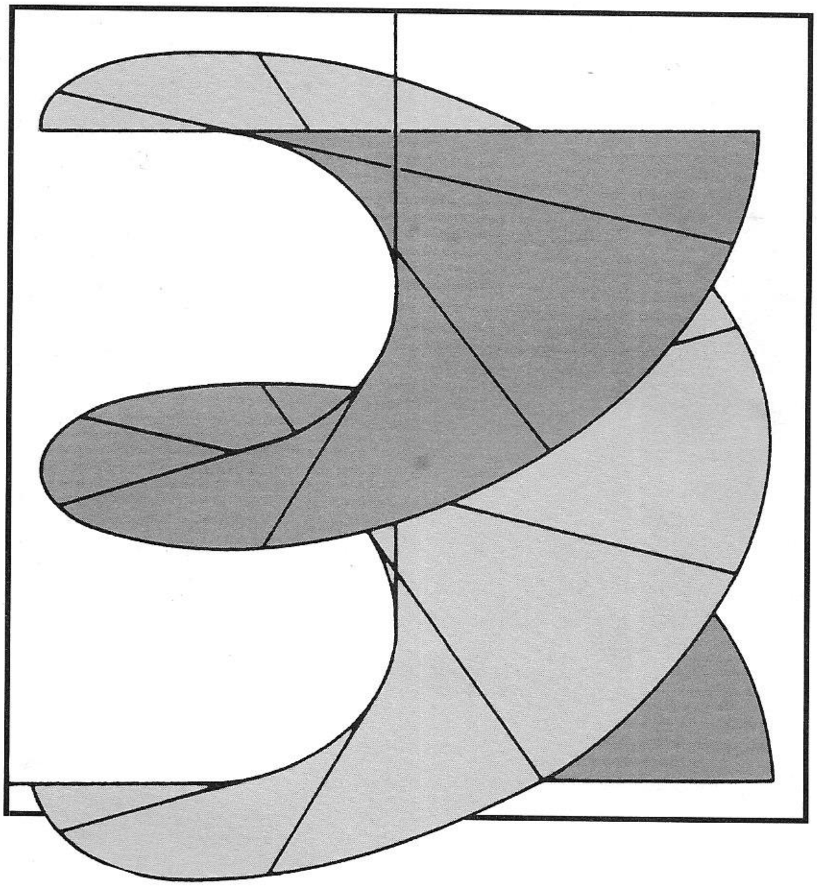
M434 Part I
Mathematics: A Fourth Level Course

M434
DIFFERENTIAL GEOMETRY



PART I
CALCULUS ON EUCLIDEAN SPACE

The Open University

The Open University

Mathematics: A Fourth Level Course

M434 Differential Geometry

Part I Calculus on Euclidean Space

Prepared for the Course Team
by Bob Margolis

Set book

Barrett O'Neill, *Elementary Differential Geometry*, hardback edition (Academic Press, 1966).

It is essential to have this book; the course is based on it and will not make sense without it.

The set book is referred to as *O'Neill*.

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published as *M334* 1976.

First published as *M434* 1993. Reprinted 1996.

Copyright © 1993 The Open University

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, without written permission from the publisher or a licence from the Copyright Licensing Agency Limited. Details of such licences (for reprographic reproduction) may be obtained from the Copyright Licensing Agency Ltd of 90 Tottenham Court Road, London, W1P 9HE.

Edited, designed and typeset by the Open University using the Open University TEX System.

Printed in the United Kingdom by Hobbs the Printers Ltd, Totton, Hampshire SO40 3WX.

ISBN 0 7492 4770 3

This text forms part of an Open University Fourth Level Course. If you would like a copy of *Studying with The Open University*, please write to the Central Enquiry Service, PO Box 200, The Open University, Walton Hall, Milton Keynes, MK7 6YZ. If you have not already enrolled on the Course and would like to buy this or other Open University material, please write to Open University Educational Enterprises Ltd, 12 Cofferidge Close, Stony Stratford, Milton Keynes, MK11 1BY, United Kingdom.

1.2

Contents

Introduction 4

1 Euclidean space 6

2 Tangent vectors 7

3 Directional derivatives 8

4 Curves 13

5 1-Forms 15

6 Differential forms 18

7 Mappings 20

8 Summary 27

Solutions to the exercises 28

Introduction

This text, together with the first two sections of *Part II*, will cover most of the mathematical ideas and methods needed in this course for studying curves and surfaces in three dimensions. Because of the concentration of methods here, unrelieved by any applications, you may find the work rather indigestible at first reading. We hope that a full appreciation of the significance of the ideas will develop as you apply them to curves and surfaces. This introduction is intended to give you some signposts to follow.

The Introduction in *O'Neill* is brief, but important, as it introduces several of the notational conventions used in the course. It also gives a brief reminder of some ideas about functions from $\mathbf{R}$ to $\mathbf{R}$.

It might be as well to say here that in the course as a whole the word 'function', when used without further qualification, should be taken to mean a function from $\mathbf{E}^3$ to $\mathbf{R}$. That is, saying 'f is a function' indicates that

$$f : \mathbf{E}^3 \longrightarrow \mathbf{R}.$$

Before we can hope to describe curves and surfaces, we need a precise description of the 'three-dimensional space' where they will be located. You have already met the vector space $\mathbf{R}^3$ as a mathematical version of the space in which we live—often referred to as (three-dimensional) *Euclidean space*. In Section 1, *O'Neill* refers to this space as $\mathbf{E}^3$. This change of name is not perversity: as will emerge, $\mathbf{E}^3$ is actually rather more than $\mathbf{R}^3$.

As mentioned in *Part 0*, in the historical note, we shall be using sets of axes (Darboux' 'moving frames') attached to each point of the object (curve or surface) that we are studying. Actually, rather than axes, we shall place a set of basis vectors at each point. In order to formalize the notion of 'vector placed at a point' the concept of *tangent vector* is introduced in Section 2. Once we have the idea of a vector placed at a particular point, we shall generalize to placing vectors at each point of $\mathbf{E}^3$ according to some rule. This gives rise to the notion of *vector field*.

These are the 'based vectors' of *MST204*.

Note: As in so much of mathematics, you have to be careful not to read too much into everyday words which are used in a new context. The 'tangent' in 'tangent vector' is a generalization of the usual meaning and does not necessarily imply that the vector is a tangent to a curve or surface in the usual sense of the word. In fact, when dealing with curves we shall have some tangent vectors which are tangents to the curve and some which are not. The new use of 'tangent' will seem strange at first, but you will become used to it with practice.

The key feature of the 'moving frames' approach is the study of the rates of change of the basis vectors as we move around the object being studied. Thus, to make any progress, we shall need careful definitions of what we mean by rates of change, or derivatives, of vectors. We do not attempt to deal with vectors immediately; we start with rates of change of functions. Here we run into the first new idea. The problem is that, in $\mathbf{E}^3$, there are infinitely many ways to set off from a particular point. The rate of change of a function may well depend not only on where you are but also in which direction you set off. Actually, we introduce a further feature: we also permit the rate of change to depend on how fast you set off, as well as in which direction. This is not actually a complication as it permits us to specify the direction and speed of setting off by a vector. The direction is specified by the direction of the vector, and the speed by the size of the vector. Taking all these factors into account leads to the concept of a *directional derivative*. Section 3 defines directional derivatives and proves that they have the linearity and Leibniz (product rule) properties that all our derivatives will have.

Section 4 introduces a formal definition of what is meant in the course by a *curve*. The definition is not quite what you might have expected; curves turn out to be *functions* rather than distorted lines in space.

In Section 5 we return to the theme of directional derivatives of functions. As we indicated above, the derivative of a function can be expected to depend on where we are (specified by three coordinates) and the vector specifying direction and speed of departure from that point (another three coordinates). Thus the directional derivative will be a function of these six quantities. To give a compact notation for such things, we introduce *1-forms.*

Having introduced 1-forms, Section 6 explores some algebra using 1-forms. Addition and subtraction turn out to be straightforward; multiplication and differentiation are more involved. The ideas from this section will not be applied to anything for some time, so you may well want to skim this section now and come back to it again later.

1-forms will be used in Sections 5–8 of *Part II.*

We are still, at this stage, some way short of the goal of definitions of derivatives of sets of basis vectors. Section 7 makes a step forward by introducing functions

$$\mathbf{E}^n \longrightarrow \mathbf{E}^m,$$

where n and m are usually 2 or 3. We refer to these as *mappings.* You will become familiar with these new objects as you work through the course. Having introduced mappings, we then investigate how they can be combined and differentiated.

Finally, Section 8 provides a very brief summary of this part.

Study advice

One possible breakdown into study weeks is the following.

Week 1 *O'Neill*, Chapter I, Introduction and Sections 1, 2 and 3.

Week 2 *O'Neill*, Chapter I, Sections 4 and 5.

Week 3 *O'Neill*, Chapter I, Sections 6, 7 and 8.

This plan leaves one study week for the first two sections of *Part II* and the associated Tutor-marked Assignment. The suggestion above is exactly that—a suggestion. After working through the first couple of sections you will have a better idea of how fast you can study *O'Neill.* Please allow for the fact that the work becomes less closely connected with previous experience as you go on, and that your progress may become slower as a result.

Do not be tempted to skip exercises. Although there is much powerful theory developed in the course, we expect you to be able to apply the theory to specific examples, and regular practice will help you. We expect that a significant part of your study time will be taken up by the exercises.

The exercises for a particular section may be taken from *O'Neill*, given in this text or both. Solutions for all exercises, regardless of origin, appear at the end of this text. Even if you feel that your solution is completely correct, please read ours; there may well be additional points made in the course of the solution.

Exercises from *O'Neill* are given by page and number.

1 Euclidean space

Read *O'Neill: Introduction and Chapter I, Section 1, pages 1–5.*

Euclidean 3-space You may well have seen this denoted by $\mathbf{R}^3$ rather than $\mathbf{E}^3$. *M203*

Natural coordinate functions This use of x, y and z to represent *functions* may well be new to you. However, the function view does correspond to how these symbols are used in elementary mathematics. For example,

'the value of x at the point $(2, 3)$ is 2'

should correspond to the sort of usage of x that is familiar. This phrase is effectively explaining what the value of the function x is at a point in its domain. All that *O'Neill* has done is to formalize this.

Differentiable In differential geometry we shall want our functions to behave 'smoothly'. If you consider a function f from $\mathbf{R}$ to $\mathbf{R}$, it will have a graph without 'sharp points' if we require that its derivative f' exists everywhere. The existence of f' on the domain of f ensures that the graph has a tangent everywhere. If we also require f' to be continuous, we shall ensure that the tangent to the graph cannot make any sudden changes of direction. Going further, requiring f to have derivatives of all possible orders should ensure that the graph of f is 'smooth'. The definition on page 4 generalizes this idea to $\mathbf{E}^3$.

The comments on the definition (page 5) simply point out that we really need to worry about f and its derivatives only on the region of $\mathbf{E}^3$ in which we are interested.

The region will always be an open subset of $\mathbf{E}^3$.

The final comment: that some of the concepts are 'over-elaborate' in one dimension is quite important. Actually, the differences between a number of the concepts apparently vanish in the one-dimensional case. Thus, in some cases, generalizing a familiar idea about functions from $\mathbf{R}$ to $\mathbf{R}$ requires not one but several new definitions.

Exercise 1.1 *O'Neill*, page 5, Exercise 1.

Exercise 1.2 *O'Neill*, page 5, Exercise 2.

Exercise 1.3 *O'Neill*, page 6, Exercise 3.

Exercise 1.4 *O'Neill*, page 6, Exercise 4. Note that this exercise defines the composite of a function

$$h : \mathbf{E}^3 \longrightarrow \mathbf{R}$$

with three functions

$$g_1, g_2, g_3 : \mathbf{E}^3 \longrightarrow \mathbf{R}.$$

[*Solutions on page 27*]

2 Tangent vectors

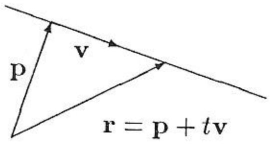
Read *O'Neill: Chapter I, Section 2, pages 6–10.*

Tangent vector The reason for the apparently curious choice of name for this concept will emerge when we study surfaces, where most of the tangent vectors will also be tangents (in the intuitive sense) to surfaces. For now, it is important to realize that a tangent vector may not be (geometrically) a tangent to anything obvious. The important thing is to visualize $\mathbf{v_p}$ as a vector parallel to $\mathbf{v}$, but with its blunt end at the point $\mathbf{p}$ rather than at the origin.

We shall be studying geometric objects from the point of view of someone living on the object, with a personal set of axes. The concept of a tangent vector is exactly what we shall need to define axes at a particular point.

We shall make another use of tangent vectors. A tangent vector $\mathbf{v_p}$ defines a line through $\mathbf{p}$ pointing in the direction of $\mathbf{v}$ by the equation

$$\mathbf{r} = \mathbf{p} + t\mathbf{v}, \quad t \in \mathbf{R}.$$



Tangent space This definition formalizes the idea of taking an egocentric view of the world! One way of looking at it is to consider the 'global' coordinate system as being imposed by an outsider. Then, if you are located at $\mathbf{p}$,

$$T_{\mathbf{p}}(\mathbf{E}^3)$$

represents your personal copy of $\mathbf{E}^3$, with its 'origin' at $\mathbf{p}$.

Vector field This concept will be used to formalize the idea of carrying a set of axes around. Instead of actually moving, say, the x-axis around, we assume that each point $\mathbf{p}$ in space has been provided with the unit vector $(1,0,0)_{\mathbf{p}}$ to indicate the direction of the (local) x-axis. The *function*

$$U_1 : \mathbf{p} \longmapsto (1,0,0)_{\mathbf{p}},$$

which provides the unit vector at each point, is an example of a vector field. Note the difference between U_1, which is a function, and $U_1(\mathbf{p})$, which is a particular tangent vector at a particular point.

Pointwise principle This principle is nothing more than the fact that to define a function you must specify its value at each point in the domain.

Natural frame field This set of vector fields essentially provides each point in $\mathbf{E}^3$ with a copy of the standard basis. Lemma 2.5 says that, because a tangent vector at any point can be expressed in terms of $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ at that point, a corresponding statement is true of a vector field.

M203

We shall discuss frame fields in general in *Part II*; for now, the only frame field that we shall use is the natural frame field. It is worth emphasizing that 'the natural frame field' means the *three* vector fields U_1,U_2 and U_3.

If you think about how actually to specify a vector field, Lemma 2.5 becomes natural, not to say inevitable. Specifying a vector field involves explaining which tangent vector should be attached to the point

$$\mathbf{p} = (p_1, p_2, p_3),$$

whatever the coordinates of $\mathbf{p}$ happen to be. Thus we would have to explain how to calculate the components of $\mathbf{v}$ from the coordinates of $\mathbf{p}$. That is, we have to specify the functions that *O'Neill* calls v_1, v_2 and v_3.

The identity

$$V = v_1U_1 + v_2U_2 + v_3U_3$$

explains the importance of the natural frame field: it forms a basis for vector fields on $\mathbf{E}^3$.

If you regard a vector field V as a way of persuading $\mathbf{E}^3$ to grow a (straight) hair at each point, then requiring V to be differentiable is to demand that the direction and size of the hairs varies smoothly as you move from point to point of $\mathbf{E}^3$.

Finally, a comment on $\mathbf{R}^3$ versus $\mathbf{E}^3$. We would regard $\mathbf{R}^3$ as a three-dimensional, real vector space and $\mathbf{E}^3$ as that vector space, *together with all its tangent vectors*.

Exercise 2.1 *O'Neill*, page 10, Exercise 1 (part (a) only).

Exercise 2.2 *O'Neill*, page 10, Exercise 2.

Exercise 2.3 *O'Neill*, page 10, Exercise 3. Your answers should be expressed in terms of x, y, z, U_1, U_2 and U_3.

Exercise 2.4 *O'Neill*, page 11, Exercise 5.

Note: This exercise shows that the natural frame field is not the only possible basis for vector fields on $\mathbf{E}^3$. The phrase *Euclidean* coordinate functions was used earlier for the v_i to indicate that they were the coordinate functions corresponding to using the *natural* frame field as a basis.

[*Solutions on page 27*]

3 Directional derivatives

Read *O'Neill: Chapter I, Section 3, pages 11–14.*

This section marks the start of the process of defining rates of change of objects (functions, tangent vectors etc.) as we move around in $\mathbf{E}^3$.

Directional derivatives There is a fundamental problem that arises in generalizing the idea of derivative from one dimension to several (here three) dimensions. When we want to find the rate of change of a function

$$f : \mathbf{R} \longrightarrow \mathbf{R}$$

at a point $a \in \mathbf{R}$, we look at the values of f at points 'near' a. A point in $\mathbf{R}$ near a can be written $a + h$, where h is either positive or negative. The derivative of f at a is found by calculating the limit of *M203*

$$\frac{f(a+h) - f(a)}{h}$$

as h tends to zero (provided that the limit exists).

Suppose now that we want the 'derivative' of a function

$$f : \mathbf{E}^3 \longrightarrow \mathbf{R}.$$

A point 'near' $\mathbf{p}$ will be of the form $\mathbf{p} + \mathbf{v}$, where $\mathbf{v}$ is a *vector*. The number h in the one-dimensional case has been replaced by the vector $\mathbf{v}$. Since division by vectors is not defined, we cannot simply translate the limit definition above into the new situation.

We have to accept that the rate of change of f will depend not only on where we are (the point $\mathbf{p}$), but also on the direction in which we move away from $\mathbf{p}$. Having accepted this change from the one-dimensional case, we make a further change: we allow the rate of change of f to depend on how 'fast' we set off from $\mathbf{p}$, as well as on the direction.

Now, the direction and speed with which we set off from $\mathbf{p}$ can conveniently be defined by giving a tangent vector $\mathbf{v}_\mathbf{p}$ based at $\mathbf{p}$, since $\mathbf{p}$ and $\mathbf{v}$ define a straight line

$$\mathbf{r} = \mathbf{p} + t\mathbf{v}, \quad t \in \mathbf{R},$$

where $t = 0$ corresponds to the point $\mathbf{p}$.

We can now define a composite function from $\mathbf{R}$ to $\mathbf{R}$ by calculating values of f along this line:

$$t \longmapsto \mathbf{p} + t\mathbf{v} \longmapsto f(\mathbf{p} + t\mathbf{v}).$$

This composite is just a function on $\mathbf{R}$, and so can be differentiated with respect to t in the usual way. The value of this (ordinary) derivative at $t = 0$ is defined to be the derivative of f with respect to the tangent vector $\mathbf{v}_\mathbf{p}$. The formal statement of the above discussion is Definition 3.1.

Note: If you prefer to denote differentiation by dashes, you may prefer the following form of the definition:

$$\mathbf{v}_\mathbf{p}[f] = (f(\mathbf{p} + t\mathbf{v}))'(0).$$

The dash represents differentiation with respect to t.

There is a strong link between the newly defined directional derivative and the partial derivatives that were discussed in *Part 0*.

Suppose that we look at the tangent vector

$$\mathbf{v}_\mathbf{p} = (1, 0, 0)_\mathbf{p}.$$

The definition of $\mathbf{v}_\mathbf{p}[f]$ now gives

$$\begin{aligned} \mathbf{v}_\mathbf{p}[f] &= (f(\mathbf{p} + t\mathbf{v}))'(0) \\ &= (f(\mathbf{p} + t(1,0,0)))'(0) \\ &= (f(p_1 + t, p_2, p_3))'(0). \end{aligned}$$

In the last line above, only the first coordinate, $p_1 + t$, is varying; the other two are constant. In other words, the last line is

$$\frac{\partial f}{\partial x}$$

evaluated at $t = 0$, in other words at $\mathbf{p}$.

Hence,

$$(1, 0, 0)_\mathbf{p}[f] = \frac{\partial f}{\partial x}(\mathbf{p}).$$

Similarly,

$$\begin{aligned} (0, 1, 0)_\mathbf{p}[f] &= \frac{\partial f}{\partial y}(\mathbf{p}), \\ (0, 0, 1)_\mathbf{p}[f] &= \frac{\partial f}{\partial z}(\mathbf{p}). \end{aligned}$$

Thus, the values of the partial derivatives at a point are the directional derivatives with respect to

$$(1, 0, 0)_\mathbf{p}, \quad (0, 1, 0)_\mathbf{p} \quad \text{and} \quad (0, 0, 1)_\mathbf{p}.$$

This link may well be what is meant by the comment in the paragraph immediately following the definition on page 11 of *O'Neill*.

The worked example following the definition shows how to calculate directional derivatives from first principles. However, Lemma 3.2 provides an alternative method which is of considerable importance.

The proof of Lemma 3.2 uses a generalized form of the chain rule (composite rule) for differentiating composite functions. You have met the rule for functions from **R** to **R**, probably in the form

M101 and *M203*

$$(f \circ g)' = f'(g) \times g'.$$

Here, we have a composite

$$t \longmapsto (p_1 + tv_1, p_2 + tv_2, p_3 + tv_3) \longmapsto f(\mathbf{p} + t\mathbf{v}).$$

If we define

$$\begin{aligned} g_1(t) &= p_1 + tv_1, \\ g_2(t) &= p_2 + tv_2, \\ g_3(t) &= p_3 + tv_3, \end{aligned}$$

then we are trying to differentiate the composite

$$f(g_1, g_2, g_3).$$

The version of the chain rule that applies in this case is

$$(f(g_1, g_2, g_3))' = \frac{\partial f}{\partial x} g_1' + \frac{\partial f}{\partial y} g_2' + \frac{\partial f}{\partial z} g_3'.$$

We shall make quite heavy use of this result. You should note it, and note also its similarity to the 'single variable' case.

It is this result that is applied to prove Lemma 3.2.

Once the definition of directional derivative and a practical method of calculation are established, *O'Neill* goes on to show that directional derivatives have the linearity and Leibnizian (product) properties that you should be familiar with from ordinary derivatives. You might like to compare parts 2 and 3 of Theorem 3.3 (*O'Neill*, page 12) with

$$\begin{aligned} (af + bg)' &= af' + bg', \\ (fg)' &= f'g + fg', \end{aligned}$$

for functions f, g from **R** to **R** and $a, b \in \mathbf{R}$.

There is no result in 'ordinary' calculus corresponding to Theorem 3.3(1) because ordinary differentiation is always carried out in a standard direction and at standard (unit) speed.

Note that the right-hand side of Theorem 3.3(3) is NOT

$$\mathbf{v_p}[f]\, g + f\, \mathbf{v_p}[g].$$

To see why not, consider what sort of object

This type of analysis will often help you to reconstruct such formulas from a general knowledge of the pattern for the Leibniz property.

$$\mathbf{v_p}[fg]$$

is. The directional derivative of a function with respect to a particular tangent vector is a real number. However,

$$\mathbf{v_p}[f]\, g + f\, \mathbf{v_p}[g]$$

is the sum of two terms, each of which is a function multiplied by a real number. Thus

$$\mathbf{v_p}[f]\, g + f\, \mathbf{v_p}[g]$$

is a function, not a real number. On the other hand, evaluating the functions in each term at $\mathbf{p}$ gives

$$\mathbf{v_p}[f]\, g(\mathbf{p}) + f(\mathbf{p})\, \mathbf{v_p}[g],$$

which is a real number.

The next step that *O'Neill* takes is similar (but not identical) to the process of going from the value of a derivative at a particular point to the idea of a derived function. Thus, in elementary calculus, after finding that the derivative of

$$f = x^2$$

Since x is a *function*, this way of defining f is permissible.

at 2 is 4, at 3 is 6 and so on, we go on to define the derived function

$$f' = 2x,$$

which gives a rule for calculating the derivative everywhere.

Our goal is to give an expression that will enable us to calculate the value of the directional derivative of a function with respect to any tangent vector at any point. In this section we go only part of the way towards this goal; the rest of the work is done in Section 5.

Instead of considering the directional derivative of a function f with respect to a single tangent vector, we now consider the directional derivatives with respect to a whole collection of tangent vectors, one at each point of $\mathbf{E}^3$. The collection of tangent vectors is provided by a vector field V. If you inspect the paragraph beginning half-way down page 13 carefully, you will see that $V[f]$ is defined as a composite function

$$\mathbf{p} \longmapsto V(\mathbf{p}) \longmapsto V(\mathbf{p})[f].$$

Since the domain of this composite is $\mathbf{E}^3$ and the codomain is $\mathbf{R}$, we have that:

> the derivative of a function with respect to a vector field is a function.

The example that immediately follows the definition is extremely important. From remarks that we made earlier,

$$(1,0,0)_{\mathbf{p}}[f] = \frac{\partial f}{\partial x}(\mathbf{p}).$$

Since $U_1(\mathbf{p}) = (1,0,0)_{\mathbf{p}}$, we have $U_1[f]$ defined as the composite

$$\mathbf{p} \longmapsto U_1(\mathbf{p}) \longmapsto (1,0,0)_{\mathbf{p}}[f] = \frac{\partial f}{\partial x}(\mathbf{p}).$$

Since the function defined by

$$\mathbf{p} \longmapsto \frac{\partial f}{\partial x}(\mathbf{p})$$

s, by definition, written

$$\frac{\partial f}{\partial x},$$

we have the result

$$U_1[f] = \frac{\partial f}{\partial x}$$

and two similar results for $U_2[f]$ and $U_3[f]$.

These results give a sound basis for thinking of the partial derivatives of a function as being the derivatives in the x, y and z directions.

Now that *O'Neill* has made the definition of $V[f]$, the next step is to show that the newly defined derivative also has the linearity and Leibnizian properties. This is the content of Corollary 3.4. Note that, this time, the Leibniz property (Corollary 3.4(3)) is the exact analogue of

$$(fg)' = f'g + fg'.$$

The results of Corollary 3.4 provide the basis for calculating directional derivatives with respect to vector fields. (Parts 1 and 2 are probably the most used.) The following worked example may help to illustrate the point.

Example Suppose that the function $f : \mathbf{E}^3 \longrightarrow \mathbf{R}$ and the vector field V on $\mathbf{E}^3$ are defined by:

$$f = xe^y, \quad V = z^2 U_1 + xz U_2 + x^3 U_3.$$

Then

$$\begin{aligned} V[f] &= (z^2 U_1 + xzU_2 + x^3 U_3)[xe^y] \\ &= z^2 U_1[xe^y] + xzU_2[xe^y] + x^3 U_3[xe^y] \quad \text{(by Corollary 3.4 (1))} \\ &= z^2 \frac{\partial}{\partial x}(xe^y) + xz\frac{\partial}{\partial y}(xe^y) + x^3 \frac{\partial}{\partial z}(xe^y) \quad (\text{using } U_1[f] = \partial f/\partial x \text{ etc.}) \\ &= z^2\, e^y + xz\, xe^y + x^3 \times 0 \\ &= ze^y(z + x^2). \end{aligned}$$

The example in *O'Neill* preceding Remark 3.5 uses exactly the same technique: application of linearity followed by

$$U_1[f] = \frac{\partial f}{\partial x}$$

etc. ■

You should note the contents of Remark 3.5, but the course texts will tend to use the full form $\mathbf{v_p}$ rather more than *O'Neill* does. In particular, we shall often retain the full form, $\mathbf{v_p}$, including the point of application $\mathbf{p}$, whereas *O'Neill* tends to retain only the vector part.

Exercise 3.1 *O'Neill*, page 14, Exercise 1.

Exercise 3.2 *O'Neill*, page 15, Exercise 2.

Exercise 3.3 The vector field V and function f are defined by

$$V = 2U_1 - U_2 + 3U_3, \quad f = e^x \cos y.$$

Calculate $V[f]$.

Exercise 3.4 *O'Neill*, page 15, Exercise 3.

Exercise 3.5 Suppose that the vector field V has Euclidean coordinate functions v_1, v_2 and v_3, so that

$$V = v_1 U_1 + v_2 U_2 + v_3 U_3.$$

Find $V[x_i]$ for each coordinate function x_i, and hence show that

$$V = \sum_{i=1}^{3} V[x_i] U_i.$$

Exercise 3.6 If V and W are two vector fields such that

$$V[f] = W[f]$$

for every real-valued function f on $\mathbf{E}^3$, show that $V = W$. (*Hint*: The result of the last exercise may be useful.)

[*Solutions on page 29*]

4 Curves

Read *O'Neill: Chapter I, Section 4, pages 15–21.*

This section formally defines curves, which is the first type of geometric object that you will study in detail in the course; the other being surfaces.

Curve The definition of curve is, perhaps, not quite what you might have expected. It is the function α which is the curve; the object in $\mathbf{E}^3$ that you probably visualize when thinking about a curve is referred to as the *route* of the curve. (The route is the image set of the function α.)

The requirement for the domain of a curve to be an interval, together with requiring a curve to be differentiable, ensure that the route of a curve is connected ('all in one piece').

Much of the language used to discuss curves is based on the idea of a moving point tracing out the route of the curve. The choice of t in

$$\alpha : t \longmapsto \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$$

suggests 'time' as the variable. We shall talk about 'speed' and 'velocity' in connection with curves.

Example 4.2 Examples 4.2(1) and 4.2(2) in *O'Neill* are particularly important and will reappear from time to time.

Example 4.2(2) The distance of a point in the xy-plane from the origin can be found by evaluating the function

$$\sqrt{x^2 + y^2}$$

at the point in question. Thus

$$(\sqrt{x^2 + y^2})(a\cos t, a\sin t, 0) = \sqrt{a^2\cos^2 t + a^2\sin^2 t} = a$$

so all points on the route of the first curve are on a circle of radius a.

Example 4.2(3) The justification for the description of the route of the curve given in *O'Neill* is contained in Exercise 1 below.

Example 4.2(5) 'Brute-force' plotting of points is only rarely helpful for curves in $\mathbf{E}^3$, partly due to the difficulty of sketching points on such routes on two-dimensional paper.

Velocity of curve This section of *O'Neill* introduces the concept of velocity and gives it a geometric interpretation. The geometric link might be a little clearer if the two diagrams on pages 17 and 18 were interchanged: the second is before the limit is taken, the first is after.

It is important to note that the velocity vector

$$(\alpha_1'(t), \alpha_2'(t), \alpha_3'(t))$$

is attached to the point $\alpha(t)$ on the route; it is a tangent vector in the sense that we have recently defined.

There is also a link with the idea of vector field. The process of finding velocities at all the points on the route of a curve amounts to defining a vector field

$$\alpha_1' U_1 + \alpha_2' U_2 + \alpha_3' U_3,$$

but *only* for points on the route of the curve.

The term 'speed' is mentioned in this section of *O'Neill* without explanation. The definition is straightforward: it is the magnitude of the velocity. Thus, at $\alpha(t)$, the speed is

$$\sqrt{(\alpha_1'(t))^2 + (\alpha_2'(t))^2 + (\alpha_3'(t))^2}.$$

Since $||\mathbf{v}||$ is used in this course to denote the size (magnitude) of the vector $\mathbf{v}$, this result can be written

You may have used $|\mathbf{v}|$ in the past.

$$\text{Speed} = ||\alpha|| = \sqrt{(\alpha_1'(t))^2 + (\alpha_2'(t))^2 + (\alpha_3'(t))^2}.$$

Reparametrization This is an important technical process, although you would be forgiven for not seeing why at this point in the course!

To give an indication of its importance, we must refer back to the method that we said would be used to study curves: each point will be provided with a 'local' basis of unit vectors and the change in this basis from place to place will give information about the shape of the curve.

The velocity vector provides *a* vector at each point on the route of the curve, but it is not necessarily of unit length. If we could arrange that the length of the velocity vector were 1 at each point (that is, the curve had unit speed everywhere), we should have a start for our basis of unit vectors. The later work on reparametrization will show that we can always arrange for the route of a curve to be traversed at unit speed.

This section makes a start by showing what happens to the velocity when the parametrization is changed. *Part II*, Section 2, will complete the process by showing how to change the parametrization to achieve unit speed.

The key step in Lemma 4.5 involves three applications of the usual chain rule formula. If you inspect the result carefully, you will see that it can be written

$$\beta = \alpha \circ h \quad \Longrightarrow \quad \beta' = \alpha'(h) \times h',$$

which is a direct analogue of the familiar form of the chain rule.

Important: Lemma 4.6 is apparently there simply because we have invented a new sort of tangent vector—velocities of curves—and we can now use such tangent vectors to differentiate functions. In a sense this is so. However, Lemma 4.6 will be of *vital* importance when we come to study surfaces, so it is worth taking some time to appreciate exactly what it says.

As *O'Neill* says on page 20, finding $\alpha'(t)[f]$ involves finding the rate of change of f along the straight line through $\alpha(t)$ in the direction of $\alpha'(t)$. The lemma shows that the rate of change of f along the tangent is the same as the rate of change along the curve itself. In one way this is not surprising: the curve and the tangent point in the same direction and the length of the tangent represents the speed of progress along the curve.

Where, as here, it causes no confusion, we say 'along the curve' rather than insist on the more precise 'along the route of the curve'.

Lemma 4.6 is of only theoretical interest at the moment. Later, however, we shall encounter functions which are defined only on a curve or surface and not on the whole of $\mathbf{E}^3$. In such cases we cannot use the definition to find directional derivatives because, given $\mathbf{v_p}$, the function in question may not be defined along the whole of the straight line $\mathbf{p} + t\mathbf{v}$. In such cases we use Lemma 4.6. If we can find a curve α, such that

$$\alpha(0) = \mathbf{p}, \quad \alpha'(0) = \mathbf{v_p},$$

then, for a function f, we have

$$\mathbf{v_p}[f] = (f(\alpha(t)))'(0).$$

Paraphrasing the result of Lemma 4.6: in the definition of directional derivative, the straight line $\mathbf{p} + t\mathbf{v}$ can be replaced by *any* curve α through $\mathbf{p}$ with velocity $\mathbf{v_p}$ at $\mathbf{p}$.

The final remarks in this section of *O'Neill* are not of great importance for this course and you need read them only cursorily.

Exercise 4.1 Let $\alpha : (0, \pi/2) \longrightarrow \mathbf{E}^3$ be the curve described in Example 4.2(3) of *O'Neill*, that is,

$$\alpha(t) = (2\cos^2 t, \sin 2t, 2\sin t), \quad 0 < t < \pi/2.$$

(a) Find the velocity vector of α for arbitrary t and also for $t = \pi/4$.

(b) Justify *O'Neill's* statements about the curve by doing the following.

  (i) Show that $\alpha(t)$ has length 2 for all values of t. (This shows that the route of α lies on the sphere of radius 2, centre the origin.)

  (ii) Show that $\alpha(t)$ is always 1 unit from the point $(1, 0, 2\sin t)$. (This shows that the route of α lies on the cylinder of radius 1 whose axis is the line $x = 1, y = 0$.)

(c) Find the coordinate functions of the curve $\beta = \alpha(h)$, where h is the function defined by

$$h(s) = \arcsin s, \quad 0 < s < 1.$$

Exercise 4.2 *O'Neill*, page 21, Exercise 4. (2)

Exercise 4.3 *O'Neill*, page 21, Exercise 5. Note that the phrase 'find a straight line ...' means 'write down the equation in the form $\alpha(t) = \ldots$ for a suitable function α'.

Exercise 4.4 *O'Neill*, page 21, Exercise 7.

Exercise 4.5 *O'Neill*, page 22, Exercise 9.

[*Solutions on page 31*]

5 1-Forms

Read *O'Neill: Chapter I, Section 5, pages 22–25.*

Erratum *O'Neill*, page 25, final line before the exercises:

for $\ldots(p_2^2 + 1)v_3$ read $\ldots(p_2^2 + 2)v_3$.

Do not worry if you have not met the definition of differential given at the start of this section.

1-forms In Section 3 we mentioned that this section would complete the task of giving a formula for calculating the directional derivative of a function at any point with respect to any tangent vector.

Consider, for a moment, what such a formula would have to look like. Given a function f and a tangent vector $\mathbf{v}_\mathbf{p}$, the directional derivative would depend on the three components p_1, p_2 and p_3 of $\mathbf{p}$ *and* on the three components v_1, v_2 and v_3 of $\mathbf{v}$.

For example, let

$$f = xyz \quad \text{and} \quad \mathbf{v_p} = (v_1, v_2, v_3)_{(p_1,p_2,p_3)},$$

then, working from Lemma3.2,

$$\begin{aligned}\mathbf{v_p}[f] &= \frac{\partial f}{\partial x}(\mathbf{p})\, v_1 + \frac{\partial f}{\partial y}(\mathbf{p})\, v_2 + \frac{\partial f}{\partial z}(\mathbf{p})\, v_3 \\ &= (yz)(\mathbf{p})\, v_1 + (xz)(\mathbf{p})\, v_2 + (xy)(\mathbf{p})\, v_3 \\ &= p_2p_3v_1 + p_1p_3v_2 + p_1p_2v_3.\end{aligned}$$

Thus, as expected, $\mathbf{v_p}[f]$ is a 'function of six variables'.

What we are aiming at is a formula (which will be called *df*) which will produce the correct value of the directional derivative when $\mathbf{v_p}$ is correctly substituted.

We already know how to produce the p_i from a point $\mathbf{p}$. We use the coordinate functions x, y and z. What we now need are functions to play a similar role in extracting the v_i from a tangent vector. That is precisely what is done by

$$dx, \quad dy \quad \text{and} \quad dz.$$

For the function defined above, we have

$$df = (yz)\, dx + (xz)\, dy + (xy)\, dz.$$

The concept of 1-form arises from the need to express directional derivatives concisely. However, they are not *defined* quite in the way that the need arises.

Because (Theorem 3.3(1)) directional derivatives behave linearly with respect to the involvement of the tangent vector, the expression that we arrive at for a general directional derivative must involve the v_i in a linear manner. Also, for a particular tangent vector, the value of the directional derivative is a real number. Thus, once we have defined it properly, *df* will have to be a function whose domain is the set of all possible tangent vectors, whose codomain is $\mathbf{R}$ and which is linear with respect to tangent vectors at each point.

If you check the example above, you will see that this is the case.

We have two properties of *df*: it is a function from tangent vectors to reals and, at any point $\mathbf{p}$, it is linear. As is so often the case in mathematics, these two properties are extracted and made the definition of a new concept. Here the concept is that of a 1-form. The formal definition is given in Definition 5.1.

Even though *df* is not properly defined yet!

Having given a definition of 1-forms (5.1) that does not mention directional derivatives, there is now the task of showing that 1-forms are the right concept for expressing general 'derivatives'. This is the point of Definition 5.2.

We have been discussing a way of representing the function

$$\mathbf{v_p} \longmapsto \mathbf{v_p}[f]$$

as a 'formula'. Definition 5.2 makes a start by giving a name, *df*, to

$$\mathbf{v_p} \longmapsto \mathbf{v_p}[f].$$

A short argument is needed to show that *df* is a 1-form. (Actually, the definition of 1-form was carefully framed so as to make *df* a 1-form.)

The argument is in the paragraph immediately after the definition.

Example 5.3(1) Having introduced a new concept, some simple examples are in order. The simplest non-constant functions on $\mathbf{E}^3$ are probably the coordinate functions. It is worth expanding the working in Example 5.3 a little. Since x is a function, we can calculate

$$\begin{aligned}\mathbf{v_p}[x] &= \frac{\partial x}{\partial x}(\mathbf{p})\, v_1 + \frac{\partial x}{\partial y}(\mathbf{p})\, v_2 + \frac{\partial x}{\partial z}(\mathbf{p})\, v_3 \\ &= 1\, v_1 + 0\, v_2 + 0\, v_3 \\ &= v_1.\end{aligned}$$

Hence

$$dx : \mathbf{v_p} \longmapsto v_1.$$

Similarly,

$$dy : \mathbf{v_p} \longmapsto v_2,$$
$$dz : \mathbf{v_p} \longmapsto v_3.$$

Thus, dx, dy and dz behave with respect to tangent vectors in exactly the same way as x, y and z behave with respect to points.

You may wonder, with justification, what this use of dx has to do with its use in expressions such as

$$\int x^2\,dx.$$

The expression after the integral sign is a 1-form (it is df, where $f = x^3/3$). If finding directional derivatives of functions yields 1-forms, it is reasonable to think that integration might produce functions from 1-forms. Not only is this reasonable but, under suitable conditions, it is also true. Unfortunately, we do not have room to consider integration of 1-forms in this course, although *O'Neill* does mention the topic briefly in several places.

Example 5.3(2) A new idea is inserted here without explicit mention. So far, 1-forms act on tangent vectors (and produce real numbers), but we can form composites of 1-forms with vector fields. The case which is used in this example involves dx, etc. with the natural frame field U_1, etc.

When we write $dx(U_1)$, we mean the composite function

$$\mathbf{p} \longmapsto U_1(\mathbf{p}) \longmapsto dx(U_1(\mathbf{p})).$$

Since $U_1(\mathbf{p}) = (1,0,0)_\mathbf{p}$ and dx extracts the first component of a tangent vector, we have

$$dx(U_1(\mathbf{p})) = 1.$$

Similar calculations show that

$$\begin{array}{lll} dx(U_1(\mathbf{p})) = 1, & dx(U_2(\mathbf{p})) = 0, & dx(U_3(\mathbf{p})) = 0, \\ dy(U_1(\mathbf{p})) = 0, & dy(U_2(\mathbf{p})) = 1, & dy(U_3(\mathbf{p})) = 0, \\ dz(U_1(\mathbf{p})) = 0, & dz(U_2(\mathbf{p})) = 0, & dz(U_3(\mathbf{p})) = 1. \end{array}$$

These nine results can be summed up in 'functional' form as

$$dx_i(U_j) = \delta_{ij}.$$

The Kronecker delta, δ_{ij}, is a standard symbol and will be used a number of times in this course.

The definition of δ_{ij} is $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

If we use the linearity properties of 1-forms and the results summarized as

$$dx_i(U_j) = \delta_{ij},$$

then evaluating 1-forms on vector fields becomes quite routine.

Example Suppose that the vector field V and the 1-form ϕ are defined by

$$V = x^2U_1 - zU_2 + yU_3 \quad \text{and} \quad \phi = y\,dx + x\,dy - z\,dz.$$

Then

$$\begin{aligned} \phi(V) &= (y\,dx + x\,dy - z\,dz)(x^2U_1 - zU_2 + yU_3) \\ &= y\,dx(x^2U_1 - zU_2 + yU_3) + x\,dy(x^2U_1 - zU_2 + yU_3) \\ &\quad - z\,dz(x^2U_1 - zU_2 + yU_3) \quad \text{(linearity)} \\ &= y(x^2)\,dx(U_1) + y(-z)\,dx(U_2) + y(y)\,dx(U_3) + x(x^2)\,dy(U_1) \\ &\quad + x(-z)\,dy(U_2) + x(y)\,dy(U_3) - z(x^2)\,dz(U_1) - z(-z)\,dz(U_2) \\ &\quad - z(y)\,dz(U_3) \quad \text{(linearity)} \\ &= (x^2y + 0 + 0) + (0 - xz + 0) + (0 + 0 - zy) \quad (\text{using } dx_i(U_j) = \delta_{ij}) \\ &= x^2y - xz - yz. \end{aligned}$$

■

Having found that dx, dy and dz provide examples of 1-forms, *O'Neill* goes on to show that these three are all the 1-forms that you need (at least for $\mathbf{E}^3$). Lemma 5.4 shows that any 1-form can be expressed in terms of these 'basic' 1-forms.

Warning: Although differentials of functions provided the motivation for introducing 1-forms and will also provide many of the examples of 1-forms, it is quite easy to write down a 1-form which is not the differential of a function on $\mathbf{E}^3$. For example, the 1-form

$$dx + x\,dy + x^2\,dz$$

cannot be df for any f. To see why not, spend a few minutes trying to find f such that

$$\frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial f}{\partial z} = x^2.$$

Calculating differentials The remark after Lemma 5.7 is useful. Using

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$

is not usually the quickest way of finding df, as the example illustrates.

Also illustrated by the example is the fact that, now differentials are available, it is easier to find a directional derivative

$$\mathbf{v}_\mathbf{p}[f]$$

by finding df and then using

$$\mathbf{v}_\mathbf{p}[f] = df(\mathbf{v}_\mathbf{p}).$$

Exercise 5.1 *O'Neill*, page 25, Exercise 1.

Exercise 5.2 *O'Neill*, page 25, Exercise 3.

Exercise 5.3 *O'Neill*, page 26, Exercise 4.

Exercise 5.4 *O'Neill*, page 26, Exercise 5 (first part only).

Exercise 5.5 *O'Neill*, page 26, Exercise 7.

[*Solutions on page 32*]

6 Differential forms

Read *O'Neill: Chapter I, Section 6, pages 26–31.*

Errata

1 *O'Neill*, page 31, Exercise 6, last line:

for $\ldots r, \phi, z \ldots$ read $\ldots r, \vartheta, z \ldots.$

2 *O'Neill*, page 29, line 8:

for $d\phi = d(xy)\,dx + d(x^2)\,dz.$ read $d\phi = d(xy) \wedge dx + d(x^2) \wedge dz.$

O'Neill is following his convention of omitting the wedge when one term is dx, dy or dz.

3 *O'Neill*, page 29, 4 lines from the bottom:

for $d\varphi$ read $d\phi$. ■

This section of *O'Neill* contains no difficult computational ideas. The problem is, rather, why introduce 2-forms and 3-forms at all? Little use is made of 3-forms in the course, but 2-forms appear in the work on surfaces.

2-forms Although it is not required for the exercises on this section, it seems reasonable to give some hints about what 2-forms *are*, in the same way as we have defined 1-forms to be real-valued functions on tangent vectors.

If you have retained a feeling, in spite of the last section, that dx has something to do with a 'small increment' in the x-direction, then you might suspect that the 2-form $dx\,dy$ might have something to do with a 'small area'. It is certainly the case that some of the 2-forms that we shall meet are connected with areas, although not all of them are.

Just as 1-forms operated on tangent vectors, 2-forms act on *pairs* of tangent vectors based at the same point. The details will emerge later in the course; for now we just give an example. The 2-form $dx\,dy$ acts on the pair of tangent vectors $(\mathbf{v_p}, \mathbf{w_p})$ at $\mathbf{p}$ as follows:

$$(dx\,dy)(\mathbf{v_p}, \mathbf{w_p}) = dx(\mathbf{v_p})\,dy(\mathbf{w_p}) - dx(\mathbf{w_p})\,dy(\mathbf{v_p}).$$

If we evaluate the terms, we obtain

$$(dx\,dy)(\mathbf{v_p}, \mathbf{w_p}) = v_1w_2 - w_1v_2.$$

The other 'basic' 2-forms $dx\,dz$ and $dy\,dz$ behave in a similar way.

The important thing at this stage is to be able to do calculations with differential forms. A more complete understanding of their meaning should emerge when we begin to use them later on.

Exterior derivative You might find it helpful to paraphrase Definition 6.3 as: 'to differentiate a 1-form, apply d to the coefficient functions'.

It is clearly possible to extend Definition 6.3 to 2-forms as follows.

Exterior derivative of 2-forms

If η is the 2-form

$$\eta = f_1\,dx\,dy + f_2\,dy\,dz + f_3\,dx\,dz,$$

then the exterior derivative of η is the 3-form

$$d\eta = df_1 \wedge dx\,dy + df_2 \wedge dy\,dz + df_3 \wedge dx\,dz.$$

Theorem 6.4 By now, the pattern used by *O'Neill* should be becoming familiar. Having defined a new form of differentiation (the exterior derivative), there follows a proof that the derivative obeys the linear and Leibniz rules that help in doing calculations. The point worthy of special note here is the form of Theorem 6.4(3) with its minus sign.

Optional Comment: If you have met the concepts of 'div', 'grad' and 'curl', you may be interested in page 31, Exercise 8, which explains the link between differential forms and these ideas.

For our purposes forms provide a very concise way of representing some of the ideas of differential geometry and we shall not use any of the other possible approaches from vector calculus.

Another, classical, approach uses tensors. We shall avoid them as well!

Exercise 6.1 *O'Neill*, page 31, Exercise 1.

Exercise 6.2 *O'Neill*, page 31, Exercise 2.

Exercise 6.3 *O'Neill*, page 31, Exercise 3.

Exercise 6.4 *O'Neill*, page 31, Exercise 4.

Exercise 6.5 *O'Neill*, page 31, Exercise 7. (*Hint*: Use the fact that any 1-form ϕ can be written as

$$\sum_{i=1}^{3} f_i \, dx_i$$

for suitable functions f_i.)

Exercise 6.6 *O'Neill*, page 32, Exercise 9.

[*Solutions on page 33*]

7 Mappings

Read *O'Neill: Chapter I, Section 7, pages 32–39.*

Errata

1 *O'Neill*, page 36, second line after the proof of Theorem 7.5:

for $F_*(v)$, read $F_*(\mathbf{v})$,

that is, the vector $\mathbf{v}$ should be in bold type.

2 *O'Neill*, page 37, proof of Corollary 7.7:

for 'in Corollary 7.6' read 'in Theorem 7.5'. ■

Mapping In this section, *O'Neill* generalizes the idea of a function

$$f : \mathbf{E}^3 \longrightarrow \mathbf{R}$$

to that of a function

$$F : \mathbf{E}^n \longrightarrow \mathbf{E}^m,$$

and if F is differentiable (in the sense defined on page 33) then such functions are called mappings. Thus, both real-valued functions on $\mathbf{E}^3$ and curves are special cases of mappings.

We shall use mappings for two purposes in this course. First, we shall use a special type of mapping (isometries) in our work on curves. Secondly, a mapping between a 'simple' surface, such as a sphere, and a more complicated one will help in the study of the shape of the more complicated surface. For this second purpose, we shall need to know how a mapping 'distorts' geometric objects (such as curves) in its domain.

Definition 7.1 This tells you how to specify a mapping: you give the coordinate functions.

If you inspect the paragraph following the definition carefully, you will see that the 'pointwise' and 'coordinate function' forms for specifying a mapping are very little different; the function form is more concise. You should find little difficulty in switching from one form to the other. What this paragraph does re-emphasize is that x, y and z are *functions*, whereas p_1, p_2 and p_3 are real numbers.

The final paragraph before Definition 7.2 makes an important point. It is not easy to decide at a glance what effect a mapping has on its domain. Just as functions from $\mathbf{R}$ to $\mathbf{R}$ can be investigated by using derivatives, so we shall use 'derivatives' of mappings to help in understanding them. To begin with, however, we look at what a mapping does to the route of a curve.

Example 7.3(1) In matrix form, with respect to the standard bases in domain and codomain, F can be written

M203

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

Example 7.3(2) There is no reason why you should have thought of the idea of considering the image of the circle α. However, it does give some insight into what the mapping does.

Definition 7.4 This is the definition of a derivative for mappings mentioned above. If we write out explicitly the calculation involved, we can see that

$$F_*(\mathbf{v_p}) = (F(\mathbf{p} + t\mathbf{v}))'(0)_{F(\mathbf{p})}.$$

In this form, it is easier to see the family connection with the definition of directional derivative of a function on $\mathbf{E}^3$. Indeed, directional derivatives can now be seen as a special case of derivative maps.

It is worth emphasizing that

> F tells you what happens to each *point* $\mathbf{p}$ in the domain;
>
> F_* tells you what happens to each *tangent vector* $\mathbf{v_p}$ at $\mathbf{p}$, by mapping it to a tangent vector at $F(\mathbf{p})$.

A very simple example might help to explain why the derivative map can give information about the distorting properties of a mapping.

Example Consider the mapping

$$F : \mathbf{E}^2 \longrightarrow \mathbf{E}^2,$$
$$F = (x + y, y),$$

and the tangent vectors

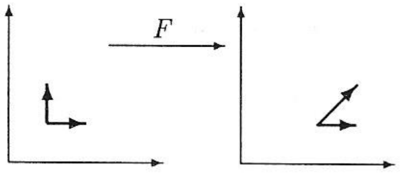
$$\mathbf{v_p} = (1, 0)_\mathbf{p},$$
$$\mathbf{w_p} = (0, 1)_\mathbf{p}.$$

Working from the definition,

$$\begin{aligned} F_*(\mathbf{v_p}) &= (F(\mathbf{p} + t(1,0)))'(0) \\ &= (F(p_1 + t, p_2))'(0) \\ &= (p_1 + t + p_2, p_2)'(0) \\ &= (1, 0)(0) \\ &= (1, 0) \quad \text{at } F(\mathbf{p}), \end{aligned}$$

$$\begin{aligned} F_*(\mathbf{w_p}) &= (F(\mathbf{p} + t(0,1)))'(0) \\ &= (F(p_1, p_2 + t))'(0) \\ &= (p_1 + p_2 + t, p_2 + t)'(0) \\ &= (1, 1)(0) \\ &= (1, 1) \quad \text{at } F(\mathbf{p}). \end{aligned}$$

Thus, at any point $\mathbf{p}$, the tangent vectors $(1, 0)$ and $(0, 1)$ are transformed to the tangent vectors $(1, 0)$ and $(1, 1)$ at the image $F(\mathbf{p})$.



This does give some idea of how the mapping distorts the domain. ■

Theorem 7.5 This result makes the connection between derivative maps and directional derivatives quite explicit. Paraphrasing: to find the value of the derivative map on $\mathbf{v_p}$, directionally differentiate the coordinate functions with respect to $\mathbf{v_p}$. (This paraphrase leaves out the point of application, but is a useful aid to memory nevertheless.)

The link with directional derivatives is the basis for a straightforward method of calculating derivative maps. The method for calculating directional derivatives in Lemma 3.2,

$$\mathbf{v_p}[f] = \frac{\partial f}{\partial x}(\mathbf{p})\, v_1 + \frac{\partial f}{\partial y}(\mathbf{p})\, v_2 + \frac{\partial f}{\partial z}(\mathbf{p})\, v_3,$$

can be written as a matrix product

$$\mathbf{v_p}[f] = \begin{pmatrix} \frac{\partial f}{\partial x}(\mathbf{p}) & \frac{\partial f}{\partial y}(\mathbf{p}) & \frac{\partial f}{\partial z}(\mathbf{p}) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

or, even,

$$\mathbf{v_p}[f] = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} (\mathbf{p}) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} .$$

Translating the result of Theorem 7.5 into matrix form, representing tangent vectors by column matrices with respect to the standard bases, leads directly to the idea of **Jacobian Matrix** discussed on page 37. For example, if

$$F = (f_1, f_2, f_3)$$

is a mapping from $\mathbf{E}^3$ to $\mathbf{E}^3$, then the image of a tangent vector $\mathbf{v_p}$ under F_* can be calculated as

$$\begin{pmatrix} \frac{\partial f_1}{\partial x}(\mathbf{p}) & \frac{\partial f_1}{\partial y}(\mathbf{p}) & \frac{\partial f_1}{\partial z}(\mathbf{p}) \\ \frac{\partial f_2}{\partial x}(\mathbf{p}) & \frac{\partial f_2}{\partial y}(\mathbf{p}) & \frac{\partial f_2}{\partial z}(\mathbf{p}) \\ \frac{\partial f_3}{\partial x}(\mathbf{p}) & \frac{\partial f_3}{\partial y}(\mathbf{p}) & \frac{\partial f_3}{\partial z}(\mathbf{p}) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

at $F(\mathbf{p})$.

It is usual to shorten

$$\begin{pmatrix} \frac{\partial f_1}{\partial x}(\mathbf{p}) & \frac{\partial f_1}{\partial y}(\mathbf{p}) & \frac{\partial f_1}{\partial z}(\mathbf{p}) \\ \frac{\partial f_2}{\partial x}(\mathbf{p}) & \frac{\partial f_2}{\partial y}(\mathbf{p}) & \frac{\partial f_2}{\partial z}(\mathbf{p}) \\ \frac{\partial f_3}{\partial x}(\mathbf{p}) & \frac{\partial f_3}{\partial y}(\mathbf{p}) & \frac{\partial f_3}{\partial z}(\mathbf{p}) \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix} (\mathbf{p}).$$

Strictly, the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix}$$

represents F_* only with respect to the standard bases in domain and codomain.

However, it is an understandable and forgiveable abuse to write

$$F_* = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial z} \end{pmatrix}$$

instead of 'F_* is represented by ... with respect to ...'.

Even more precisely, we should say that the matrix represents F_* only with respect to the natural frame field in domain and codomain. We should do so because F_* transforms tangent vectors, not vectors.

Linear approximation (Comment in second paragraph at top of page 37.) This comment refers to the link between the approximation

$$f(a+h) \approx f(a) + f'(a) \times h$$

for differentiable functions from $\mathbf{R}$ to $\mathbf{R}$ and the approximation

$$F(\mathbf{p}+\mathbf{v}) \approx F(\mathbf{p}) + F_*(\mathbf{v}).$$

The parallel is more obvious when the second approximation is written in column coordinate form:

$$F(\mathbf{p}+\mathbf{v}) \approx F(\mathbf{p}) + (\text{Jacobian matrix at } \mathbf{p}) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Theorem 7.8 This result is important not only because it says that velocities of curves are preserved by F_* but also because it is the analogue of Lemma 4.6. It says that the straight line

$$\mathbf{p} + t\mathbf{v}$$

in the definition of F_* can be replaced by any curve through $\mathbf{p}$ with velocity $\mathbf{v}_\mathbf{p}$. Since F_* is obtained by directionally differentiating the coordinate functions, you should expect results that are true of directional derivatives to be true, in a suitable sense, of derivative maps.

Definition 7.9 and Theorem 7.10 These are mentioned here for completeness. In this course we shall use this definition and result in only one case. When we define surfaces, we shall use a two-dimensional version of the definition of curve. We shall have mappings from a suitable subset of $\mathbf{E}^2$ to $\mathbf{E}^3$. Although the surface defined by such mappings may be a very complicated object when viewed as a whole, we shall want any small piece of it to look quite like a piece of the plane. Regularity of the mappings used to define surfaces is exactly the condition that we need.

Checking regularity involves finding the rank of the Jacobian matrix. Now, the rank of a matrix is the maximum number of linearly independent rows (or columns). In this course, we are interested in mappings from $\mathbf{E}^n$ to $\mathbf{E}^m$ in the cases

It is a theorem of linear algebra that for *any* matrix the maximum number of linearly independent rows is the same as the maximum number of linearly independent columns.

$$n, m = 1, 2, 3.$$

If the dimension of the domain is greater than that of the codomain, then the mapping cannot be regular. For example, if

$$F : \mathbf{E}^3 \longrightarrow \mathbf{E}^2$$
$$F = (f_1, f_2),$$

then the Jacobian matrix representing F_* is

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \end{pmatrix}(\mathbf{p}).$$

This cannot have more than 2 independent rows, so its rank cannot exceed 2, which is less than the dimension of the domain.

In practice, the commonest situation in which you will be required to check for regularity is for mappings

$$\mathbf{E}^2 \longrightarrow \mathbf{E}^3,$$

where the task reduces to showing that the 3×2 Jacobian matrix has two independent columns.

Composite mappings An important result is hidden by *O'Neill* in two exercises. Because of the importance of the result, we ask you to do the two exercises now. Our solutions (with comments) follow immediately.

Try *O'Neill*, page 40, Exercises 7 and 12(b) now.

O'Neill writes GF for the composite, where you may be used to writing $G \circ F$.

Solution to Exercise 7 In full:

$$\begin{aligned} G(F(p_1, p_2)) &= G(f_1(p_1, p_2), f_2(p_1, p_2)) \\ &= (g_1(f_1(p_1, p_2), f_2(p_1, p_2)), g_2(f_1(p_1, p_2), f_2(p_1, p_2))). \end{aligned}$$

In function form:

$$GF = (g_1(f_1, f_2), g_2(f_1, f_2)).$$

Because F and G are mappings, their coordinate functions are differentiable. Thus, the coordinate functions of GF involve composites of differentiable functions. Hence, the coordinate functions of GF are differentiable and GF is a mapping.

Solution to Exercise 12(b) This requires three applications of Theorem 7.8. First, let

$$\beta = G(F(\alpha)).$$

Then, by Theorem 7.8 applied to GF

$$(GF)_*(\alpha') = \beta'.$$

Now, if we define

$$\gamma = F(\alpha)$$

and apply Theorem 7.8 to F, we obtain

$$F_*(\alpha') = \gamma'.$$

But

$$\beta = G(\gamma),$$

so

$$G_*(\gamma') = \beta'.$$

Putting these results together gives

$$\begin{aligned} (GF)_*(\alpha') &= \beta' \\ &= G_*(\gamma') \\ &= G_*(F_*(\alpha')). \end{aligned}$$

The consequence of this last result is a 'composite rule' for derivative maps. Since

$$(GF)_*(\alpha') = G_*(F_*(\alpha'))$$

holds for *all* possible curves in the domain, we conclude that

$$(GF)_* = G_*(F_*).$$

The composite rule as stated above is a little dangerous when using Jacobian matrices to carry out calculations. If we recall the composite rule for functions from $\mathbf{R}$ to $\mathbf{R}$ in the form

$$(g \circ f)' = g'(f) \times f',$$

perhaps you can see why. The derivative g' must be evaluated at the image under the first function, f.

Consider what happens under the various composites that we are discussing. The mappings transform points:

$$\mathbf{p} \longmapsto F(\mathbf{p}) \longmapsto G(F(\mathbf{p})).$$

The derivative maps transform tangent vectors:

$$\mathbf{v}_{\mathbf{p}} \longmapsto F_*(\mathbf{v}_{\mathbf{p}})_{F(\mathbf{p})} \longmapsto (GF)_*(\mathbf{v}_{\mathbf{p}})_{G(F(\mathbf{p}))}.$$

It should be clear that F_* can be achieved by applying the value of the Jacobian matrix for F at $\mathbf{p}$ but that G_* requires the value of the Jacobian matrix for G at $F(\mathbf{p})$.

This last statement suggests that a 'safer' statement of the composite rule for mappings is

$$(G \circ F)_* = G_*(F)F_*,$$

where $G_*(F)$ means G_* evaluated at F.

This alternative form of the composite rule for maps has the advantage of being very similar to the one for ordinary functions.

The similarity may make it more memorable.

The set of exercises for this section includes practice at finding derived maps of composites using Jacobians.

Exercise 7.1 Express each of the following mappings from $\mathbf{E}^3$ to $\mathbf{E}^3$ in terms of the Euclidean coordinate functions.

(a) $F : \mathbf{p} \longmapsto -3\mathbf{p}$ (b) $F : \mathbf{p} \longmapsto (e^{p_1 p_2}, p_3 + 2p_2, p_1^2)$

Exercise 7.2 The mapping $F : \mathbf{E}^2 \longrightarrow \mathbf{E}^2$ is defined by

$$F = (2xy, x^2 - y^2).$$

Find the images under F of the following lines, and identify the image curves.

(a) $x = 1$ (b) $y = -1$

Exercise 7.3 Let $F : \mathbf{E}^3 \longrightarrow \mathbf{E}^3$ be the mapping

$$F = (x - y, x + y, 2z),$$

from *O'Neill*, Example 7.3.

(a) Find, from first principles, the derivative map F_*.

(b) Show that F is linear, in the sense that

$$F(a\mathbf{p} + b\mathbf{q}) = aF(\mathbf{p}) + bF(\mathbf{q}), \quad \text{for all } \mathbf{p}, \mathbf{q} \in \mathbf{E}^3,\ a, b \in \mathbf{R}.$$

(c) Generalize the result for F to show that if G is *any* linear transformation (mapping) from $\mathbf{E}^3$ to $\mathbf{E}^3$, then

$$G_*(\mathbf{v}_{\mathbf{p}}) = G(\mathbf{v})_{G(\mathbf{p})}.$$

Exercise 7.4 Let F be a mapping from $\mathbf{E}^3$ to $\mathbf{E}^3$ and let G be a mapping from $\mathbf{E}^3$ to $\mathbf{E}^2$ defined by

$$F = (x + y, x - 2y, x + y + z) \quad \text{and} \quad G = (x^2 + y^2, y^2 + z^2).$$

(a) Write down the Jacobian matrix for F_*.

(b) Write down the Jacobian matrix for G_* and evaluate it at $F(x, y, z)$.

(c) Use the composite rule to find the Jacobian matrix for $(G \circ F)_*$.

[*Solutions on page 35*]

8 Summary

Read *O'Neill: Chapter I, Section 8, page 41.*

The first chapter of *O'Neill* has not provided all the tools needed for our study of curves. We shall need a little more discussion of frame fields and some work on reparametrization. The first two sections of *Part II* will provide what is necessary.

There is one 'spin-off' from the work on derivative maps that we have done. We can provide a general version of the chain rule. Suppose that we extend the work done on Exercise 7, page 40 of *O'Neill*.

We have mappings

$$F = (f_1, f_2), \quad G = (g_1, g_2)$$

from $\mathbf{E}^2$ to $\mathbf{E}^2$. The corresponding Jacobians are

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_1}{\partial y} \\ \dfrac{\partial g_2}{\partial x} & \dfrac{\partial g_2}{\partial y} \end{pmatrix}.$$

The composite mapping $G \circ F$ has Jacobian

$$\begin{pmatrix} \dfrac{\partial g_1(f_1, f_2)}{\partial x} & \dfrac{\partial g_1(f_1, f_2)}{\partial y} \\ \dfrac{\partial g_2(f_1, f_2)}{\partial x} & \dfrac{\partial g_2(f_1, f_2)}{\partial y} \end{pmatrix}.$$

Writing out the composite rule for derived maps gives

$$\begin{pmatrix} \dfrac{\partial g_1(f_1, f_2)}{\partial x} & \dfrac{\partial g_1(f_1, f_2)}{\partial y} \\ \dfrac{\partial g_2(f_1, f_2)}{\partial x} & \dfrac{\partial g_2(f_1, f_2)}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_1}{\partial y} \\ \dfrac{\partial g_2}{\partial x} & \dfrac{\partial g_2}{\partial y} \end{pmatrix} (F) \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix}.$$

Taking the top left element of both sides gives

$$\frac{\partial g_1(f_1, f_2)}{\partial x} = \frac{\partial g_1}{\partial x}(f_1, f_2)\frac{\partial f_1}{\partial x} + \frac{\partial g_1}{\partial y}(f_1, f_2)\frac{\partial f_2}{\partial x}.$$

There are three other, similar results.

A similar approach for a function of three variables would give

$$\frac{\partial g_1(f_1, f_2, f_3)}{\partial x} = \frac{\partial g_1}{\partial x}(f_1, f_2, f_3)\frac{\partial f_1}{\partial x} + \frac{\partial g_1}{\partial y}(f_1, f_2, f_3)\frac{\partial f_2}{\partial x} + \frac{\partial g_1}{\partial z}(f_1, f_2, f_3)\frac{\partial f_3}{\partial x}.$$

You might like to show that the composite rule for derived maps reduces to the ordinary chain rule in the case of mappings (functions) from $\mathbf{R}$ to $\mathbf{R}$. (*Hint*: A 1×1 Jacobian matrix is just like a function.)

Solutions to the exercises

Solution 1.1

(a) We have

$$fg^2 = (x^2y)(y\sin z)^2 = x^2y^3\sin^2 z.$$

(b) Here

$$\frac{\partial f}{\partial x} = 2xy,$$

$$\frac{\partial g}{\partial y} = \sin z,$$

$$\begin{aligned}\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial y}f &= (2xy)(y\sin z) + (\sin z)(x^2y)\\ &= xy(2y+x)\sin z.\end{aligned}$$

(c) Note that, by definition,

$$\frac{\partial^2(fg)}{\partial y\partial z} = \frac{\partial}{\partial y}\frac{\partial(fg)}{\partial z}.$$

We differentiate first with respect to z, then with respect to y. Although it is perfectly easy to find fg and work on that, we shall indicate how to use the product rule:

$$\begin{aligned}\frac{\partial(fg)}{\partial z} &= \frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z}\\ &= 0\times g + f\times y\cos z\\ &= x^2y^2\cos z,\\ \frac{\partial^2(fg)}{\partial y\partial z} &= \frac{\partial(x^2y^2\cos z)}{\partial y}\\ &= 2x^2y\cos z.\end{aligned}$$

(d) This provides an example suitable for using the chain (composite) rule:

$$\begin{aligned}\frac{\partial\sin(f)}{\partial y} &= \frac{d(\sin(f))}{df}\frac{\partial f}{\partial y}\\ &= \cos(f)\,x^2\\ &= x^2\cos(x^2y).\end{aligned}$$

Note that we could have written the start of this solution as

$$\ldots\sin'(f)\frac{\partial f}{\partial y} = \ldots.$$

Solution 1.2

(a) This solution is full to the point of tediousness just to illustrate what is meant by the pointwise principle (abbreviated here to pp).

$$\begin{aligned}&f(1,1,1) = (x^2y - y^2z)(1,1,1)\\ &= (x^2y)(1,1,1) - (y^2z)(1,1,1) \quad \text{(pp for subtraction)}\\ &= (x^2)(1,1,1)y(1,1,1) - (y^2)(1,1,1)z(1,1,1)\\ &\qquad\qquad\qquad\qquad \text{(pp for multiplication)}\\ &= x(1,1,1)x(1,1,1)y(1,1,1) - y(1,1,1)y(1,1,1)z(1,1,1)\\ &\qquad\qquad\qquad\qquad \text{(pp for multiplication)}\\ &= 1\times1\times1 - 1\times1\times1 = 0.\end{aligned}$$

In practice, you probably did the calculation with a fraction of the effort expended here.

(b) This time we give a shorter solution!

$$\begin{aligned}f(3,-1,1/2) &= (x^2y - y^2z)(3,-1,1/2)\\ &= 3^2\times(-1) - (-1)^2\times(1/2)\\ &= -9 - 1/2 = -19/2.\end{aligned}$$

(c) Here

$$f(a,1,1-a) = a^2\times1 - 1^2\times(1-a) = a^2 + a - 1.$$

(d) This time

$$f(t,t^2,t^3) = t^2\times t^2 - (t^2)^2t^3 = t^4 - t^7.$$

Solution 1.3

These solutions use the linearity properties, the Leibniz property and the chain rule as appropriate.

(a) Here

$$\begin{aligned}\frac{\partial f}{\partial x} &= \sin(xy) + x\frac{\partial\sin(xy)}{\partial x} + 0\cos(xz) + y\frac{\partial\cos(xz)}{\partial x}\\ &= \sin(xy) + xy\cos(xy) - yz\sin(xz).\end{aligned}$$

(b) In this case

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{d(\sin(g))}{dg}\frac{\partial g}{\partial x} \quad \text{(chain rule)}\\ &= \cos g\frac{d(e^h)}{dh}\frac{\partial h}{\partial x} \quad \text{(chain rule)}\\ &= \cos g\, e^h\, 2x\\ &= 2xe^{x^2+y^2+z^2}\cos\left(e^{x^2+y^2+z^2}\right).\end{aligned}$$

Solution 1.4

What would be extremely useful for this question is a generalized chain rule. However, for now, we calculate the composite function in each case and differentiate directly.

(a) Here

$$f = (x+y)^2 - (y^2)(x+z),$$

so

$$\frac{\partial f}{\partial x} = 2(x+y) - y^2(1+0) = 2(x+y) - y^2.$$

(b) This time

$$\begin{aligned}f &= e^{2x} - e^{x+y}e^x\\ &= e^{2x} - e^{2x}e^y\\ &= e^{2x}(1-e^y),\end{aligned}$$

so

$$\frac{\partial f}{\partial x} = 2e^{2x}(1-e^y).$$

(c) Now

$$f = x^2 - (-x)(x) = 2x^2,$$

so

$$\frac{\partial f}{\partial x} = 4x.$$

Solution 2.1

By definition, the vector part of $3\mathbf{v}_\mathbf{p} - 2\mathbf{w}_\mathbf{p}$ is

$$\begin{aligned}3\mathbf{v} - 2\mathbf{w} &= 3(-2,1,-1) - 2(0,1,3)\\ &= (-6,1,-9).\end{aligned}$$

Hence

$$3\mathbf{v}_\mathbf{p} - 2\mathbf{w}_\mathbf{p} = (-6,1,-9)_\mathbf{p}.$$

This result could also be written

$$3\mathbf{v}_\mathbf{p} - 2\mathbf{w}_\mathbf{p} = -6U_1(\mathbf{p}) + U_2(\mathbf{p}) - 9U_3(\mathbf{p}).$$

Solution 2.2

First, we find $W - xV$;

$$\begin{aligned} W - xV &= (2x^2U_2 - U_3) - x(xU_1 + yU_2) \\ &= 2x^2U_2 - U_3 - x^2U_1 - xyU_2 \\ &= -x^2U_1 + (2x^2 - xy)U_2 - U_3. \end{aligned}$$

Now, the value at $\mathbf{p} = (-1, 0, 2)$ of this vector field is

$$\begin{aligned} &(-x^2U_1 + (2x^2 - xy)U_2 - U_3)(-1,0,2) \\ =& -(-1)^2U_1(-1,0,2) + (2(-1)^2 - (-1)(0))U_2(-1,0,2) \\ &- U_3(-1,0,2) \\ =& -U_1(-1,0,2) + 2U_2(-1,0,2) - U_3(-1,0,2) \\ =& -(1,0,0)_{(-1,0,2)} + 2(0,1,0)_{(-1,0,2)} - (0,0,1)_{(-1,0,2)} \\ =& (-1,2,-1)_{(-1,0,2)}. \end{aligned}$$

Solution 2.3

(a) Since

$$7V = 2z^2U_1 - xyU_3,$$

we have

$$V = \frac{2z^2}{7}U_1 + 0U_2 - \frac{xy}{7}U_3.$$

You may, quite reasonably, have omitted the middle term since it is zero.

(b) We can rewrite the information given as

$$V(\mathbf{p}) = p_1U_1(\mathbf{p}) + (p_3 - p_1)U_2(\mathbf{p}) + 0U_3(\mathbf{p}).$$

If we also use $x(\mathbf{p}) = p_1$, etc., we obtain

$$V(\mathbf{p}) = x(\mathbf{p})U_1(\mathbf{p}) + (z(\mathbf{p}) - x(\mathbf{p}))U_2(\mathbf{p}) + 0U_3(\mathbf{p})$$

which, by the pointwise principle, defines V to be

$$V = xU_1 + (z - x)U_2 + 0U_3.$$

(c) We have $V = xU_1 + 2yU_2 + xy^2U_3$.

(d) First, we need the vector part of $V(\mathbf{p})$, which is

$$(1 + p_1, p_2p_3, p_2) - (p_1, p_2, p_3) = (1, p_2p_3 - p_2, p_2 - p_3).$$

By an analysis like that in the first part of this question, we obtain

$$V = U_1 + y(z-1)U_2 + (y - z)U_3.$$

(e) Here the vector part is $(-p_1, -p_2, -p_3)$, so

$$V(\mathbf{p}) = (-p_1, -p_2, -p_3)_{\mathbf{p}}$$

and so

$$V = -xU_1 - yU_2 - zU_3.$$

Solution 2.4

(a) By the definition of independence of tangent vectors, it is enough to show that the vector parts are linearly independent. The vector parts are

$$(1, 0, -p_1), \quad (0, 1, 0) \quad \text{and} \quad (p_1, 0, 1).$$

We apply the standard test for linear independence. Suppose that

$$a(1,0,-p_1) + b(0,1,0) + c(p_1,0,1) = \mathbf{0}, \quad a, b, c \in \mathbf{R}.$$

Comparing components gives

$$\begin{aligned} a + cp_1 &= 0, \\ b &= 0, \\ -ap_1 + c &= 0. \end{aligned}$$

We have $b = 0$ immediately. The third equation gives $c = ap_1$, and substituting in the first gives

$$a + ap_1^2 = a(1 + p_1^2) = 0.$$

Since $1 + p_1^2 > 0$, we have $a = 0$ and hence $c = 0$. Thus

$$a = b = c = 0,$$

and the three vectors are linearly independent.

(b) We want

$$\begin{aligned} xU_1 + yU_2 + zU_3 &= fV_1 + gV_2 + hV_3 \\ &= f(U_1 - xU_3) + gU_2 + h(xU_1 + U_3), \end{aligned}$$

where f, g and h are functions from $\mathbf{E}^3$ to $\mathbf{R}$. Comparing the coefficients of the U_i gives

$$\begin{aligned} x &= f + hx, \\ y &= g, \\ z &= -fx + h. \end{aligned}$$

We have

$$g = y,$$

and substituting $f = x - hx$ from the first equation into the third gives

$$\begin{aligned} z &= -(x - hx)x + h \\ &= h(1 + x^2) - x^2. \end{aligned}$$

Hence

$$h = (x^2 + z)/(1 + x^2)$$

and

$$\begin{aligned} f &= x - \frac{x^2 + z}{1 + x^2}x \\ &= x(1 - \frac{x^2 + z}{1 + x^2}) \\ &= \frac{x(1 - z)}{1 + x^2}. \end{aligned}$$

Hence

$$xU_1 + yU_2 + zU_3 = \frac{x(1-z)}{1 + x^2}V_1 + yV_2 + \frac{x + z}{1 + x^2}V_3.$$

Solution 3.1

To apply the definition we shall need $\mathbf{p} + t\mathbf{v}$:

$$\mathbf{p} + t\mathbf{v} = (2 + 2t, -t, -1 + 3t).$$

(a) Using the above:

$$\begin{aligned} \mathbf{v}_{\mathbf{p}}[f] &= ((y^2z)(2 + 2t, -t, -1 + 3t))'(0) \\ &= ((-t)^2(-1 + 3t))'(0) \\ &= (2t(-1 + 3t) + t^2(3))(0) \quad \text{(product rule)} \\ &= 0 + 0 = 0. \end{aligned}$$

(b) Working as in the last part,

$$\begin{aligned} \mathbf{v}_{\mathbf{p}}[f] &= ((2 + 2t)^7)'(0) \\ &= (7(2 + 2t)^6 \times 2)(0) \\ &= 7 \times 2^6 \times 2 \\ &= 896. \end{aligned}$$

(c) This time

$$\begin{aligned} \mathbf{v}_{\mathbf{p}}[f] &= (e^{2+2t}\cos(-t))'(0) \\ &= (2e^{2+2t}\cos(-t) + e^{2+2t}(-\sin(-t))(-1))(0) \\ &= 2e^2\cos 0 + e^2 \sin 0 \\ &= 2e^2. \end{aligned}$$

Solution 3.2

For each part, we calculate the three partial derivatives, evaluate at $\mathbf{p}$ and then apply the formula from Lemma 3.2.

(a) $$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial x}(\mathbf{p}) = 0,$$
$$\frac{\partial f}{\partial y} = 2yz, \quad \frac{\partial f}{\partial y}(\mathbf{p}) = 2(0)(-1) = 0,$$
$$\frac{\partial f}{\partial z} = y^2, \quad \frac{\partial f}{\partial z}(\mathbf{p}) = 0^2 = 0,$$
$$\mathbf{v_p}[f] = 0 \times 2 + 0 \times (-1) + 0 \times 3 = 0.$$

(b) $$\frac{\partial f}{\partial x} = 7x^6, \quad \frac{\partial f}{\partial x}(\mathbf{p}) = 7 \times 2^6 = 448,$$
$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y}(\mathbf{p}) = 0,$$
$$\frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial z}(\mathbf{p}) = 0,$$
$$\mathbf{v_p}[f] = 448(2) + 0 + 0 = 896.$$

(c) $$\frac{\partial f}{\partial x} = e^x \cos y, \quad \frac{\partial f}{\partial x}(\mathbf{p}) = e^2 \cos 0 = e^2,$$
$$\frac{\partial f}{\partial y} = -e^x \sin y, \quad \frac{\partial f}{\partial y}(\mathbf{p}) = -e^2 \sin 0 = 0,$$
$$\frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial z}(\mathbf{p}) = 0,$$
$$\mathbf{v_p}[f] = e^2(2) + 0 + 0 = 2e^2.$$

Solution 3.3

The main technique used here is $U_i[f] = \partial f/\partial x_i$. We also use linearity but, as usual, this is more or less taken for granted.

$$\begin{aligned}
V[f] &= (2U_1 - U_2 + 3U_3)[f] \\
&= 2U_1[f] - U_2[f] + 3U_3[f] \quad \text{(linearity)} \\
&= 2\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} + 3\frac{\partial f}{\partial z} \\
&= 2e^x \cos y - e^x(-\sin y) + 3(0) \\
&= e^x(2\cos y + \sin y).
\end{aligned}$$

Solution 3.4

These questions are all tackled by applying the various parts of Corollary 3.4 (pages 13–14) and $U_i[f] = \partial f/\partial x_i$.

(a) Here
$$\begin{aligned}
V[f] &= (y^2 U_1 - xU_3)[f] \\
&= y^2 U_1[f] - xU_3[f] \quad \text{(Corollary 3.4(1))} \\
&= y^2 \frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z} \\
&= y^2(y) - x(0) \\
&= y^3.
\end{aligned}$$

(b) This time,
$$\begin{aligned}
V[g] &= y^2 U_1[g] - xU_3[g] \\
&= y^2 \frac{\partial g}{\partial x} - x\frac{\partial g}{\partial z} \\
&= y^2(0) - x(3z^2) \\
&= -3xz^2.
\end{aligned}$$

(c) Now
$$\begin{aligned}
V[fg] &= V[f]\,g + f\,V[g] \quad \text{(Corollary 3.4(3))} \\
&= y^3 g + f\,(-3xz^2) \quad \text{(using parts (i) and (ii))} \\
&= y^3(z^3) + xy(-3xz^2) \\
&= yz^2(y^2 z - 3x^2).
\end{aligned}$$

(d) In this case,
$$\begin{aligned}
fV[g] - gV[f] &= xy(-3xz^2) - z^3(y^3) \\
&= -yz^2(3x^2 + y^2 z).
\end{aligned}$$

(e) We could do this question very simply by calculating $f^2 + g^2$.

However, there is some merit in seeing how Corollary 3.4 gives an extension to a familiar result from ordinary calculus. First, note
$$\begin{aligned}
V[f^2] &= V[ff] \\
&= V[f]\,f + f\,V[f] \\
&= 2f\,V[f].
\end{aligned}$$
This result may remind you of applying the (ordinary) chain rule to obtain
$$(f^2)' = 2ff'.$$
Applying the result above and a corresponding one for $V[g^2]$,
$$\begin{aligned}
V[f^2 + g^2] &= V[f^2] + V[g^2] \quad \text{(linearity)} \\
&= 2f\,V[f] + 2g\,V[g] \\
&= 2(xy)(y^3) + 2(z^3)(-3xz^2) \\
&= 2x(y^4 - 3z^5).
\end{aligned}$$

(f) Now, we have
$$\begin{aligned}
V[V[f]] &= V[y^3] \\
&= (y^2 U_1 - xU_3)[y^3] \\
&= y^2 U_1[y^3] - xU_3[y^3] \\
&= y^2 \frac{\partial y^3}{\partial x} - x\frac{\partial y^3}{\partial z} \\
&= y^2(0) - x(0) \\
&= 0.
\end{aligned}$$

Solution 3.5

The question directs us to calculate the three derivatives
$$V[x_i] = v_1 U_1[x_i] + v_2 U_2[x_i] + v_3 U_3[x_i], \quad i = 1, 2, 3.$$
To do this, note that
$$\frac{\partial x_1}{\partial x_1} = 1, \quad \frac{\partial x_1}{\partial x_2} = 0, \quad \text{etc.}$$
That is,
$$\frac{\partial x_i}{\partial x_i} = 1, \quad \frac{\partial x_i}{\partial x_j} = 0, \quad i \neq j.$$
Hence
$$V[x_i] = v_1 U_1[x_i] + v_2 U_2[x_i] + v_3 U_3[x_i] = v_i, \quad i = 1, 2, 3.$$
This is because only the term whose suffix is i is non-zero. Hence, by the definition of V,
$$\begin{aligned}
V &= v_1 U_1 + v_2 U_2 + v_3 U_3 \\
&= V[x_1]U_1 + V[x_2]U_2 + V[x_3]U_3 \\
&= \sum_{i=1}^{3} V[x_i]U_i.
\end{aligned}$$

Solution 3.6

The method of solving this question illustrates a good general point. When some property is given to hold for *all* of some class of objects, it is always worth applying the information to some 'easy' particular cases. Here we are told that $V[f] = W[f]$ for all real-valued functions f on $\mathbf{E}^3$. We may obtain useful information by applying this to the coordinate functions x_i, $i = 1, 2, 3$. We shall use the standard convention that

$$V = v_1U_1 + v_2U_2 + v_3U_3$$
$$W = w_1U_1 + w_2U_2 + w_3U_3.$$

By what we know,

$$V[x_i] = W[x_i], \quad i = 1, 2, 3,$$

but we also know that

$$V[x_i] = v_i, \; W[x_i] = w_i,$$

so

$$v_i = w_i, \quad i = 1, 2, 3.$$

This means that $V = W$.

Solution 4.1

(a) We obtain the velocity by differentiating each coordinate function:

$$\alpha'(t) = (-4\cos t \sin t, 2\cos 2t, 2\cos t)_{\alpha(t)}, \quad 0 < t < \frac{\pi}{2}.$$

Note that the velocity vector has a point of application since it is a tangent vector (as well as being a vector tangent to the curve).

Evaluating at $\pi/4$, we use

$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},$$
$$\cos\frac{\pi}{2} = 0, \quad \sin\frac{\pi}{2} = 1.$$

Thus

$$\alpha'\left(\frac{\pi}{4}\right) = (-2, 0, \sqrt{2})_{(1,1,\sqrt{2})}.$$

(b) (i) The length of $\alpha(t)$ is given by

$$\begin{aligned}
\|\alpha(t)\| &= \sqrt{4\cos^4 t + \sin^2 2t + 4\sin^2 t} \\
&= \sqrt{4\cos^4 t + 4\sin^2 t\cos^2 t + 4\sin^2 t} \\
&\quad (\sin 2t = 2\sin t\cos t) \\
&= \sqrt{4\cos^2 t(\cos^2 t + \sin^2 t) + 4\sin^2 t} \\
&= \sqrt{4\cos^2 t + 4\sin^2 t} \\
&= \sqrt{4} = 2.
\end{aligned}$$

(ii) By similar methods,

$$\begin{aligned}
\|\alpha(t) - (1, 0, 2\sin t)\| &= \sqrt{(2\cos^2 t - 1)^2 + \sin^2 2t + 0^2} \\
&= \sqrt{\cos^2 2t + \sin^2 2t} \\
&\quad (\cos 2t = 2\cos^2 t - 1) \\
&= \sqrt{1} = 1.
\end{aligned}$$

(c) Given that

$$t = h(s) = \arcsin s,$$

that is,

$$s = \sin t,$$

we shall need $\cos t$ in terms of s as well:

$$\cos t = \sqrt{1 - \sin^2 t} = \sqrt{1 - s^2}.$$

Hence, using

$$\sin 2t = 2\sin t\cos t,$$

we have

$$\begin{aligned}
\beta(s) &= \alpha(h(s)) \\
&= (2(\sqrt{1-s^2})^2, 2s\sqrt{1-s^2}, 2s) \\
&= (2(1-s^2), 2s\sqrt{1-s^2}, 2s).
\end{aligned}$$

Solution 4.2

If we assume that the coordinate functions of α are α_1, α_2 and α_3, then we know that

$$\alpha_1(0) = 1, \quad \alpha_1'(t) = t^2,$$
$$\alpha_2(0) = 0, \quad \alpha_2'(t) = t,$$
$$\alpha_3(0) = -5, \quad \alpha_3'(t) = e^t.$$

Integrating, and using the initial values, gives

$$\alpha_1(t) = \frac{t^3}{3} + 1,$$
$$\alpha_2(t) = \frac{t^2}{2},$$
$$\alpha_3(t) = e^t - 5,$$

so

$$\alpha(t) = \left(\frac{t^3}{3} + 1, \frac{t^2}{2}, e^t - 5\right).$$

Solution 4.3

We begin by expressing the equation of each line in

$$\mathbf{p} + t\mathbf{v}$$

form. For the first line,

$$\mathbf{p} = (1, -3, -1),$$
$$\mathbf{v} = (6, 2, 1) - (1, -3, -1) = (5, 5, 2).$$

Hence the equation is

$$\begin{aligned}
\alpha(t) &= (1, -3, -1) + t(5, 5, 2) \\
&= (1 + 5t, -3 + 5t, -1 + 2t).
\end{aligned}$$

For the second line,

$$\mathbf{p} = (-1, 1, 0),$$
$$\mathbf{v} = (-5, -1, -1) - (-1, 1, 0) = (-4, -2, -1).$$

This gives

$$\beta(s) = (-1 - 4s, 1 - 2s, -s).$$

In order to discover if these lines meet, we must try to solve

$$(1 + 5t, -3 + 5t, -1 + 2t) = (-1 - 4s, 1 - 2s, -s).$$

Comparing the third components shows that, if there is a solution, we must have $s = 1 - 2t$. The first component now gives

$$1 + 5t = -1 - 4s = -1 - 4(1 - 2t) = 8t - 5,$$

so

$$6 = 3t$$

or

$$t = 2.$$

Hence

$$s = 1 - 4 = -3.$$

Checking the second components:

$$-3 + 5t = -3 + 5(2) = 7,$$
$$1 - 2s = 1 - (-6) = 7.$$

Thus, taking $t = 2$, $s = -3$ shows that the point $(11, 7, 3)$ lies on both lines.

Solution 4.4

(a) The three velocity vectors are

$$(1, 2t, 1)_{(t, 1+t^2, t)},$$
$$(\cos t, -\sin t, 1)_{(\sin t, \cos t, t)} \quad \text{and}$$
$$(\cosh t, \sinh t, 1)_{(\sinh t, \cosh t, t)}.$$

It follows that the velocity vectors at $t = 0$ are all

$$\mathbf{v}_P = (1, 0, 1)_{(0,1,0)}.$$

(b) We apply the corollary to Lemma 4.6 that was discussed earlier in this section. For each curve, we evaluate f on the curve, differentiate and then substitute $t = 0$:

$$f(t, 1 + t^2, t) = t^2 - (1 + t^2)^2 + t^2,$$
$$\text{Derivative} = 2t - 2(1 + t^2)(2t) + 2t,$$
$$\text{At } t = 0, \text{ value} = 0 - 0 + 0.$$

Hence $\mathbf{v}_P[f] = 0$.

Similarly with the second curve:

$$f(\sin t, \cos t, t) = \sin^2 t - \cos^2 t + t^2,$$
$$\text{Derivative} = 2 \sin t \cos t + 2 \cos t \sin t + 2t,$$
$$\text{At } t = 0 \text{ value} = 0 + 0 + 0 = 0.$$

For the third:

$$f(\sinh t, \cosh t, t) = \sinh^2 t - \cosh^2 t + t^2,$$
$$\text{Derivative} = 2 \sinh t \cosh t - 2 \cosh t \sinh t + 2t,$$
$$\text{At } t = 0 \text{ value} = 0 + 0 + 0 = 0.$$

Solution 4.5

Following the definition given in the question, we first find the velocity vector:

$$\alpha'(t) = (-2 \sin t, 2 \cos t, 1)_{(2 \cos t, 2 \sin t, t)}.$$

Now we write down the tangent line:

$$u \longmapsto (2 \cos t, 2 \sin t, t) + u(-2 \sin t, 2 \cos t, 1),$$

that is,

$$u \longmapsto (2(\cos t - u \sin t), 2(\sin t + u \cos t), t + u).$$

Hence, at $t = 0$ we obtain

$$u \longmapsto (2, 2u, u).$$

At $t = \pi/4$, the tangent line is

$$u \longmapsto (\sqrt{2}(1 - u), \sqrt{2}(1 + u), \pi/4 + u).$$

Note: In the above solution we have opted to find the general form of the tangent line and then substitute to obtain the two special cases asked for. You may well have dealt with each special case individually.

Solution 5.1

(a) We use the properties of dx, etc., that is

$$dx(\mathbf{v}_P) = v_1, \quad dy(\mathbf{v}_P) = v_2, \quad dz(\mathbf{v}_P) = v_3.$$

We have

$$\begin{aligned}(y^2\, dx)((1, 2, -3)_{(0,-2,1)}) &= y^2(0, -2, 1)\, dx(1, 2, -3)\\ &= (-2)^2(1)\\ &= 4.\end{aligned}$$

(b) Similarly,

$$\begin{aligned}&(z\, dy - y\, dz)((1, 2, -3)_{(0,-2,1)})\\ &= z(0, -2, 1) dy(1, 2, -3) - y(0, -2, 1) dz(1, 2, -3)\\ &= 1(2) - (-2)(-3)\\ &= -4.\end{aligned}$$

(c) Here,

$$\begin{aligned}&((z^2 - 1)dx - dy + x^2\, dz)((1, 2, -3)_{(0,-2,1)})\\ &= (1^2 - 1)(1) - (2) + 0^2(-3)\\ &= 0 - 2 + 0\\ &= -2.\end{aligned}$$

Solution 5.2

In this solution we make heavy use of the values of dx, dy and dz on U_1, U_2 and U_3:

$$\begin{aligned}&dx(U_1) = 1, \quad dx(U_2) = 0, \quad dx(U_3) = 0,\\ &dy(U_1) = 0, \quad dy(U_2) = 1, \quad dy(U_3) = 0,\\ &dz(U_1) = 0, \quad dz(U_2) = 0, \quad dz(U_3) = 1.\end{aligned}$$

(a) Expanding ϕ and using linearity gives

$$\begin{aligned}\phi(V) &= (x^2\, dx - y^2\, dz)(xU_1 + yU_2 + zU_3)\\ &= x^2\, dx(xU_1 + yU_2 + zU_3) - y^2\, dz(xU_1 + yU_2 + zU_3)\\ &= x^2(x\, dx(U_1) + y\, dx(U_2) + z\, dx(U_3))\\ &\quad - y^2(x\, dz(U_1) + y\, dz(U_2) + z\, dz(U_3))\\ &\quad \text{(linearity of 1-forms)}\\ &= x^2(x + 0 + 0) - y^2(0 + 0 + z)\\ &= x^3 - y^2 z.\end{aligned}$$

(b) Similarly,

$$\begin{aligned}\phi(W) &= (x^2\, dx - y^2\, dz)((xy + yz)U_1 - yzU_2 - xyU_3)\\ &= x^2(xy + yz) - y^2(-xy)\\ &\quad \text{(method as first solution)}\\ &= xy(x^2 + xz - y^2).\end{aligned}$$

(c) There is no need to start from scratch with this one, we can use the two previous results and linearity:

$$\begin{aligned}\phi\left(\frac{1}{x}V + \frac{1}{y}W\right) &= \frac{1}{x}\phi(V) + \frac{1}{y}\phi(W) \quad \text{(linearity)}\\ &= \frac{1}{x}(x^3 - y^2 z) + \frac{1}{y}xy(x^2 + xz - y^2)\\ &= \frac{x^3 - y^2 z}{x} + x(x^2 + xz - y^2).\end{aligned}$$

Solution 5.3

These are all applications of the chain rule.

(a) Here,

$$d(f^5) = 5f^4\, df.$$

(b) This time,

$$\begin{aligned}d(\sqrt{f}) &= \frac{1}{2}f^{-1/2}\, df\\ &= \frac{df}{2\sqrt{f}}.\end{aligned}$$

(c) Finally,

$$\begin{aligned}d(\log(1 + f^2)) &= \frac{1}{1 + f^2} 2f\, df\\ &= \frac{2f\, df}{1 + f^2}.\end{aligned}$$

Solution 5.4

It is worth noting that the three partial derivatives of

$$(x^2 + y^2 + z^2)^{1/2}$$

will all have the same form because of the symmetry:

$$\begin{aligned}\frac{\partial(x^2 + y^2 + z^2)^{1/2}}{\partial x} &= \tfrac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x)\\ &= \frac{x}{(x^2 + y^2 + z^2)^{1/2}}.\end{aligned}$$

Similarly,

$$\frac{\partial(x^2 + y^2 + z^2)^{1/2}}{\partial y} = \frac{y}{(x^2 + y^2 + z^2)^{1/2}},$$

$$\frac{\partial(x^2 + y^2 + z^2)^{1/2}}{\partial z} = \frac{z}{(x^2 + y^2 + z^2)^{1/2}}.$$

Hence

$$df = \frac{1}{(x^2+y^2+z^2)^{1/2}}(x\,dx + y\,dy + z\,dz).$$

Solution 5.5

In each case we check the two requirements: that ϕ be a real-valued function on tangent vectors and that it be linear at each point of $\mathbf{E}^3$.

In order to express the 1-forms in terms of dx, dy and dz, we use

$$dx(\mathbf{v}) = v_1, \quad dy(\mathbf{v}) = v_2, \quad dz(\mathbf{v}) = v_3$$

and the pointwise principle.

As usual, we use dx etc. and dx_1 etc. interchangeably.

(a) This is a 1-form. First, by the definition given, ϕ is a real-valued function on tangent vectors. Next,

$$\begin{aligned}\phi(a\mathbf{v}+b\mathbf{w}) &= (av_1+bw_1)-(av_3+bw_3)\\ &= a(v_1-v_3)+b(w_1-w_3)\\ &= a\phi(\mathbf{v})+b\phi(\mathbf{w}),\end{aligned}$$

so ϕ is linear at each point of $\mathbf{E}^3$.

We have

$$\phi(\mathbf{v}) = v_1 - v_3 = dx(\mathbf{v}) - dz(\mathbf{v}),$$

so, by the pointwise principle,

$$\phi = dx - dz.$$

(b) This is not a 1-form, although it is a real-valued function on tangent vectors. It fails to be linear, as the following counterexample shows. Let $\mathbf{v}$ be any tangent vector at the point $\mathbf{p} = (1,0,0)$. Then

$$\phi(\mathbf{v}) = 1 - 0 = 1.$$

However, $\phi(2\mathbf{v}) = 1 - 0 = 1$, since $2\mathbf{v}$ is based at the same point, thus

$$\phi(2\mathbf{v}) \neq 2\phi(\mathbf{v}).$$

(c) This is a 1-form. It is a real-valued function on tangent vectors and, at each point, p_1 and p_3 are fixed so

$$\begin{aligned}\phi(a\mathbf{v}+b\mathbf{w}) &= (av_1+bw_1)p_3+(av_2+bw_2)p_1\\ &= a(v_1p_3+v_2p_1)+b(w_1p_3+w_2p_1)\\ &= a\phi(\mathbf{v})+b\phi(\mathbf{w}),\end{aligned}$$

so ϕ is linear.

Here

$$\begin{aligned}\phi(\mathbf{v}) &= v_1p_3 + v_2p_1\\ &= dx(\mathbf{v})\,z(\mathbf{p}) + dy(\mathbf{v})\,x(\mathbf{p}),\end{aligned}$$

so

$$\phi = z\,dx + x\,dy.$$

(d) This is a 1-form. By the definition, $\mathbf{v_p}[f]$ is a real number for any function f on $\mathbf{E}^3$, so ϕ is a real-valued function on tangent vectors. Also, Theorem 3.3(1) shows that

$$\begin{aligned}\phi(a\mathbf{v}+b\mathbf{w}) &= (a\mathbf{v}+b\mathbf{w})[x^2+y^2]\\ &= a\mathbf{v}[x^2+y^2]+b\mathbf{w}[x^2+y^2]\\ &= a\phi(\mathbf{v})+b\phi(\mathbf{w}),\end{aligned}$$

and ϕ is linear at each point.

We first evaluate $\phi(\mathbf{v})$ using

$$\mathbf{v_p}[f] = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}(\mathbf{p})\,v_i$$

as

$$\begin{aligned}\phi(\mathbf{v}) &= \mathbf{v_p}[x^2+y^2]\\ &= 2x(\mathbf{p})\,v_1 + 2y(\mathbf{p})\,v_2 + 0\,v_3\\ &= 2x(\mathbf{p})\,dx(\mathbf{v}) + 2y(\mathbf{p})\,dy(\mathbf{v}).\end{aligned}$$

Thus

$$\phi = 2x\,dx + 2y\,dy.$$

Note: You could simply have quoted *O'Neill*'s general result that df is a 1-form whose value on $\mathbf{v_p}$ is $\mathbf{v_p}[f]$. The last part follows from

$$df = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}\,dx_i.$$

(e) This is a 1-form. It is a real-valued function on tangent vectors. Also,

$$\begin{aligned}\phi(a\mathbf{v}+b\mathbf{w}) &= 0\\ &= 0+0\\ &= a\phi(\mathbf{v})+b\phi(\mathbf{w}).\end{aligned}$$

In this case it is clear that

$$\phi = 0\,dx + 0\,dy + 0\,dz.$$

(f) This is not a 1-form. It fails to be linear, and the same counterexample as used in part (b) will illustrate this failure. (Use any tangent vector based at $(1,0,0)$.)

Solution 6.1

In the first part we make use of the distributive properties of the wedge product, the alternation rule and the 'repeats are zero' property.

In the second part, we use the definition of exterior derivative.

(a) (i) We have

$$\begin{aligned}\phi\wedge\psi &= (yz\,dx + dz)\wedge(\sin z\,dx + \cos z\,dy)\\ &= yz\sin z\,dx\,dx + yz\cos z\,dx\,dy\\ &\quad + \sin z\,dz\,dx + \cos z\,dz\,dy\\ &= yz\cos z\,dx\,dy - \sin z\,dx\,dz - \cos z\,dy\,dz.\end{aligned}$$

(ii) Here

$$\begin{aligned}\psi\wedge\xi &= (\sin z\,dx + \cos z\,dy)\wedge(dy + z\,dz)\\ &= \sin z\,dx\,dy + z\sin z\,dx\,dz\\ &\quad + \cos z\,dy\,dy + z\cos z\,dy\,dz\\ &= \sin z\,dx\,dy + z\sin z\,dx\,dz + z\cos z\,dy\,dz.\end{aligned}$$

(iii) In this case

$$\begin{aligned}\xi\wedge\phi &= (dy + z\,dz)\wedge(yz\,dx + dz)\\ &= yz\,dy\,dx + dy\,dz + yz^2\,dz\,dx + z\,dz\,dz\\ &= -yz\,dx\,dy - yz^2\,dx\,dz + dy\,dz.\end{aligned}$$

(b) We use the definition, which amounts to saying 'to find the exterior derivative, find d of the coefficient functions'.

(i) We obtain

$$\begin{aligned}d\phi &= d(yz\,dx + dz)\\ &= d(yz)\,dx + d(1)\,dz\\ &= (0\,dx + z\,dy + y\,dz)dx + 0\\ &= z\,dy\,dx + y\,dz\,dx\\ &= -z\,dx\,dy - y\,dx\,dz.\end{aligned}$$

(ii) Here

$$\begin{aligned} d\psi &= d(\sin z\,dx + \cos z\,dy) \\ &= d(\sin z)\,dx + d(\cos z)\,dy \\ &= \cos z\,dz\,dx - \sin z\,dz\,dy \\ &= -\cos z\,dx\,dz + \sin z\,dy\,dz. \end{aligned}$$

(iii) In this case

$$\begin{aligned} d\xi &= d(dy + z\,dz) \\ &= d(1)\,dy + dz\,dz \\ &= 0. \end{aligned}$$

Solution 6.2

We calculate the three terms individually. First,

$$\begin{aligned} d\,(\phi \wedge \psi) &= d\left(\frac{1}{y}\,dx \wedge z\,dy\right) \\ &= d\left(\frac{z}{y}\,dx\,dy\right) \\ &= d\left(\frac{z}{y}\right)\,dx\,dy \\ &= \left(\frac{-z}{y^2}\,dy + \frac{1}{y}\,dz\right)\,dx\,dy \\ &= \frac{-z}{y^2}\,dy\,dx\,dy + \frac{1}{y}\,dz\,dx\,dy \\ &= 0 - \frac{1}{y}\,dx\,dz\,dy \\ &= -\left(-\frac{1}{y}\,dx\,dy\,dz\right) \\ &= \frac{1}{y}\,dx\,dy\,dz. \end{aligned}$$

Next,

$$\begin{aligned} d\phi \wedge \psi &= d\left(\frac{1}{y}\,dx\right) \wedge z\,dy \\ &= -\frac{1}{y^2}\,dy\,dx \wedge z\,dy \\ &= 0 \quad \text{because } dy \text{ is repeated.} \end{aligned}$$

Finally,

$$\begin{aligned} -\phi \wedge d\psi &= -\frac{1}{y}\,dx \wedge d(z\,dy) \\ &= -\frac{1}{y}\,dx \wedge dz\,dy \\ &= -\frac{1}{y}\,dx\,dz\,dy \\ &= \frac{1}{y}\,dx\,dy\,dz. \end{aligned}$$

Comparing these results, we see that

$$d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$$

in this case.

Solution 6.3

This solution hinges on the fact that 'mixed' second derivatives are independent of the order of differentiation (at least for the well-behaved functions considered in this course). That is,

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

We know that

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz.$$

So, finding the differential of each coefficient,

$$\begin{aligned} d(df) &= \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial x}\,dx + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\,dy + \frac{\partial}{\partial z}\frac{\partial f}{\partial x}\,dz\right)\,dx \\ &\quad + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\,dx + \frac{\partial}{\partial y}\frac{\partial f}{\partial y}\,dy + \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\,dz\right)\,dy \\ &\quad + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial z}\,dx + \frac{\partial}{\partial y}\frac{\partial f}{\partial z}\,dy + \frac{\partial}{\partial z}\frac{\partial f}{\partial z}\,dz\right)\,dz \\ &= \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\,dy\,dx + \frac{\partial}{\partial x}\frac{\partial f}{\partial y}\,dx\,dy \\ &\quad + \frac{\partial}{\partial z}\frac{\partial f}{\partial x}\,dz\,dx + \frac{\partial}{\partial x}\frac{\partial f}{\partial z}\,dx\,dz \\ &\quad + \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\,dz\,dy + \frac{\partial}{\partial y}\frac{\partial f}{\partial z}\,dy\,dz \\ &= 0 + 0 + 0 = 0. \end{aligned}$$

The final collapse is caused by the remark before the solution and the alternation rule.

Applying the above result gives

$$\begin{aligned} d(f\,dg) &= df \wedge dg + f\,d(dg) \quad \text{(Theorem 6.4(2))} \\ &= df \wedge dg + 0 \\ &= df \wedge dg. \end{aligned}$$

Solution 6.4

In this solution we apply the various parts of Theorem 6.4 together with linearity and the alternation rule. The result from the last solution is also useful.

(a) We obtain

$$\begin{aligned} d(f\,dg + g\,df) &= df \wedge dg + f\,d(dg) + dg \wedge df + g\,d(df) \\ &= df \wedge dg + 0 + dg \wedge df + 0 \\ &= df \wedge dg + dg \wedge df \\ &= 0. \end{aligned}$$

Note: If you happen to recognize that

$$f\,dg + g\,df = d(fg),$$

then the result can be obtained quickly:

$$d(f\,dg + g\,df) = d(d(fg)) = 0.$$

(b) In this case, we have

$$\begin{aligned} & d((f-g)(df+dg)) \\ &= d(f-g) \wedge (df+dg) + (f-g)d(df+dg) \\ &= (df-dg) \wedge (df+dg) + (f-g)(d(df)+d(dg)) \\ &= df \wedge df + df \wedge dg - dg \wedge df - dg \wedge dg + (f-g)(0+0) \\ &= df \wedge dg + df \wedge dg \\ &= 2\,df \wedge dg. \end{aligned}$$

(c) Here we apply Theorem 6.4(3), then Theorem 6.4(2):

$$\begin{aligned} d(f\,dg \wedge g\,df) &= d(f\,dg) \wedge (g\,df) - (f\,dg) \wedge d(g\,df) \\ &= (df \wedge dg + f\,d(dg)) \wedge (g\,df) - ((f\,dg) \wedge (dg \wedge df + g\,d(df))) \\ &= (df \wedge dg + 0) \wedge (g\,df) - (f\,dg) \wedge (dg \wedge df + 0) \\ &= 0. \end{aligned}$$

(Each term has a repeated differential.)

(d) Here we have

$$\begin{aligned} & d(gf\,df) + d(f\,dg) \\ &= d(gf) \wedge df + gf\,d(df) + df \wedge dg + f\,d(dg) \\ &= (f\,dg + g\,df) \wedge df + 0 + df \wedge dg + 0 \\ &= f\,dg \wedge df + 0 + df \wedge dg \\ &= (1-f)\,df \wedge dg. \end{aligned}$$

Solution 6.5

Following the hint, if we start with

$$\phi = f_1\,dx_1 + f_2\,dx_2 + f_3\,dx_3,$$

then

$$d\phi = df_1 \wedge dx_1 + df_2 \wedge dx_2 + df_3 \wedge dx_3,$$

by the definition of exterior product. Now, consider the exterior derivative of each of the terms on the right-hand side of the last equation. For the ith term,

$$\begin{aligned} d(df_i \wedge dx_i) &= d(df) \wedge dx_i - df \wedge d(dx_i) \\ &= 0 + 0 = 0. \end{aligned}$$

Both terms are zero because $d(dg) = 0$ for any function g and so the second derivatives of both f and x_i are zero. Hence $d(d\phi) = 0$.

Solution 6.6

Because the functions are $\mathbf{E}^2 \longrightarrow \mathbf{R}$, we have the differentials

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy,$$

$$dg = \frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy.$$

Hence

$$\begin{aligned} df \wedge dg &= (\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy) \wedge (\frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy) \\ &= 0 + \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\,dx\,dy + \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\,dy\,dx + 0 \\ &= (\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x})\,dx\,dy \\ &= \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{vmatrix}\,dx\,dy. \end{aligned}$$

Note: This result does imply the alternation rule because $dg \wedge df$ will give rise to a determinant consisting of the same entries as above but with the columns interchanged. As you can easily verify, interchanging two columns of a determinant changes the sign. Hence

$$dg \wedge df = -df \wedge dg.$$

Solution 7.1

We have used x, y, z for the coordinate functions. Answers expressed in terms of x_1 etc. are equally acceptable.

(a) Since

$$-3\mathbf{p} = (-3p_1, -3p_2, -3p_3),$$

we have

$$F = (-3x, -3y, -3z).$$

(b) Here we have

$$F = (e^{xy}, z + 2y, x^2).$$

Solution 7.2

(a) If $\mathbf{p}$ lies on the line $x = 1$, then $\mathbf{p} = (1, p_2)$. Thus

$$F(\mathbf{p}) = (2p_2, 1 - p_2^2).$$

There are various ways of deciding what curve in E^2 is represented by

$$p_2 \longmapsto (2p_2, 1 - p_2^2).$$

One way is to see that the coordinates are related by

$$y = 1 - \left(\frac{x}{2}\right)^2,$$

and so the (route of the) curve is a parabola.

(b) A similar approach shows that if $\mathbf{p} = (p_1, -1)$, then

$$F(\mathbf{p}) = (-2p_1, p_1^2 - 1),$$

so the curve has equation

$$y = \left(\frac{-x}{2}\right)^2 - 1 = \left(\frac{x}{2}\right)^2 - 1.$$

This is also a parabola.

Solution 7.3

(a) Suppose that $\mathbf{v_p}$ is a tangent vector to $\mathbf{E}^3$. Then, by the definition,

$$F_*(\mathbf{v_p}) = (F(\mathbf{p} + t\mathbf{v}))'(0)_{F(\mathbf{p})}.$$

Now,

$$\begin{aligned} F(\mathbf{p} + t\mathbf{v}) &= F(p_1 + tv_1, p_2 + tv_2, p_3 + tv_3) \\ &= ((p_1 + tv_1) - (p_2 + tv_2), (p_1 + tv_1) + (p_2 + tv_2), 2(p_3 + tv_3)) \\ &= ((p_1 - p_2) + t(v_1 - v_2), (p_1 + p_2) + t(v_1 + v_2), 2p_3 + 2tv_3) \end{aligned}$$

$$(F(\mathbf{p} + t\mathbf{v}))' = (v_1 - v_2, v_1 + v_2, 2v_3),$$

$$(F(\mathbf{p} + t\mathbf{v}))'(0) = (v_1 - v_2, v_1 + v_2, 2v_3),$$

$$F(\mathbf{p}) = (p_1 - p_2, p_1 + p_2, 2p_3).$$

Hence

$$F_*(\mathbf{v_p}) = (v_1 - v_2, v_1 + v_2, 2v_3)_{(p_1 - p_2, p_1 + p_2, 2p_3)}.$$

Note that *in this case* we also have

$$F_*(\mathbf{v_p}) = F(\mathbf{v})_{F(\mathbf{p})}.$$

(b) Using the standard notation for components of $\mathbf{p}$ and $\mathbf{q}$, we have

$$\begin{aligned} F(a\mathbf{p} + b\mathbf{q}) &= F(ap_1 + bq_1, ap_2 + bq_2, ap_3 + bq_3) \\ &= ((ap_1 + bq_1) - (ap_2 + bq_2), (ap_1 + bq_1) + (ap_2 + bq_2), \\ &\qquad 2(ap_3 + bq_3)) \\ &= (a(p_1 - p_2) + b(q_1 - q_2), a(p_1 + p_2) + b(q_1 + q_2), a2p_3 + b2q_3) \\ &= a(p_1 - p_2, p_1 + p_2, 2p_3) + b(q_1 - q_2, q_1 + q_2, 2q_3) \\ &= aF(\mathbf{p}) + bF(\mathbf{q}). \end{aligned}$$

(c) We apply the definition of G_* and the information that G is linear:

$$\begin{aligned} G_*(\mathbf{v_p}) &= (G(\mathbf{p} + t\mathbf{v}))'(0)_{G(\mathbf{p})} \\ &= (G(\mathbf{p}) + tG(\mathbf{v}))'(0)_{G(\mathbf{p})} \quad \text{(linearity)} \\ &= (0 + G(\mathbf{v}))(0)_{G(\mathbf{p})} \quad \text{(differentiating w.r.t. } t\text{)} \\ &= G(\mathbf{v})_{G(\mathbf{p})}. \end{aligned}$$

Solution 7.4

(a) Since

$$\frac{\partial(x + y)}{\partial x} = 1$$

etc., we have that the Jacobian for F is

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

(b) Similarly, the Jacobian for G is

$$\begin{pmatrix} 2x & 2y & 0 \\ 0 & 2y & 2z \end{pmatrix}.$$

Evaluating this at F gives

$$\begin{pmatrix} 2(x + y) & 2(x - 2y) & 0 \\ 0 & 2(x - 2y) & 2(x + y + z) \end{pmatrix}$$

since $x(F) = x(x+y, x-2y, x+y+z) = x+y$, etc.

(c) Combining the last two parts gives the Jacobian for $G \circ F$ as

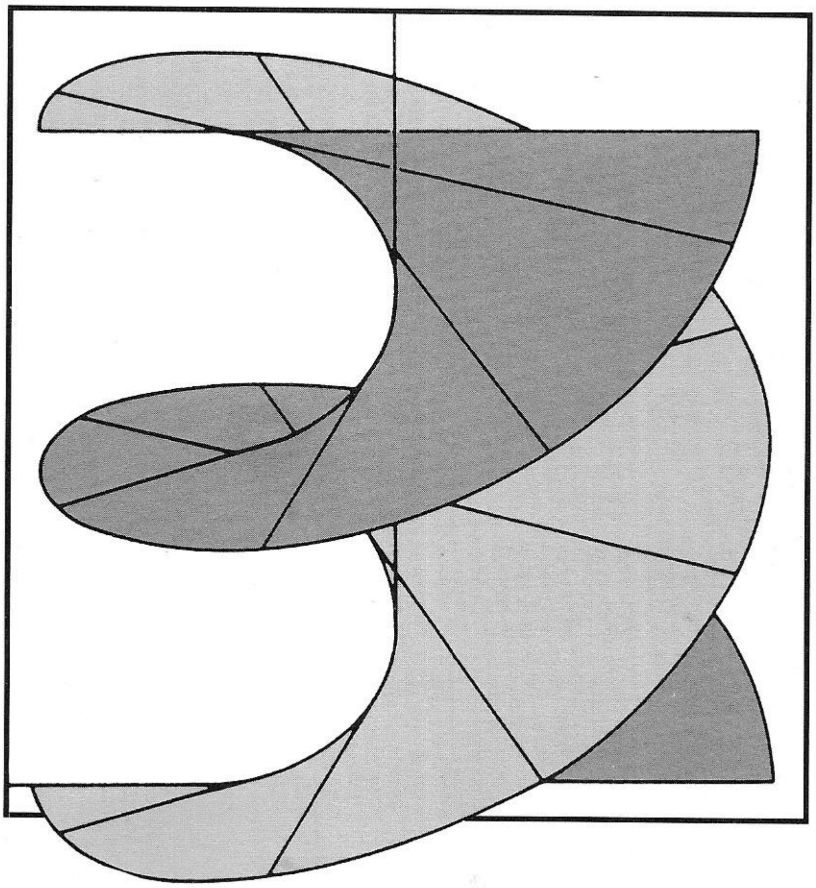
$$\begin{pmatrix} 2(x+y) & 2(x-2y) & 0 \\ 0 & 2(x-2y) & 2(x+y+z) \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2(2x-y) & 2(-x+5y) & 0 \\ 2(2x-y+z) & 2(-x+5y+z) & 2(x+y+z) \end{pmatrix}.$$

ISBN 0 7492 4770 3

M434 Part II
Mathematics: A Fourth Level Course

M434
DIFFERENTIAL GEOMETRY



PART II
FRAME FIELDS AND CURVES

The Open University

The Open University

Mathematics: A Fourth Level Course

M434 Differential Geometry

Part II Frame Fields and Curves

Prepared for the Course Team
by Bob Margolis

Set book

Barrett O'Neill, *Elementary Differential Geometry*, hardback edition (Academic Press, 1966).

It is essential to have this book; the course is based on it and will not make sense without it.

The set book is referred to as *O'Neill*.

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published as *M334* 1976.

Copyright © 1993 The Open University

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, without written permission from the publisher or a licence from the Copyright Licensing Agency Limited. Details of such licences (for reprographic reproduction) may be obtained from the Copyright Licensing Agency Ltd of 90 Tottenham Court Road, London, W1P 9HE.

Edited, designed and typeset by the Open University using the Open University TEX System.

ISBN 0 7492 4771 1

This text forms part of an Open University Fourth Level Course. If you would like a copy of *Studying with The Open University*, please write to the Central Enquiry Service, PO Box 200, The Open University, Walton Hall, Milton Keynes, MK7 6YZ. If you have not already enrolled on the Course and would like to buy this or other Open University material, please write to Open University Educational Enterprises Ltd, 12 Cofferidge Close, Stony Stratford, Milton Keynes, MK11 1BY, United Kingdom.

Contents

Introduction 4

1 Dot product 5

2 Curves 7

3 The Frenet formulas 8

4 Arbitrary-speed curves 12

5 Covariant derivatives 14

6 Frame fields 17

7 Connection forms 18

8 The structural equations 20

9 Summary 24

Solutions to the exercises 25

Introduction

This part of the course begins the study of curves, the first application of the ideas introduced in Part I.

The second application, to surfaces, begins in Part IV.

There are a few geometric preliminaries which are needed and these are dealt with in Sections 1 and 2. *O'Neill* assumes that the idea of dot and cross product of vectors is understood already; what is done now is formally to transfer these operations to tangent vectors. We shall need these operations for investigating curves because we shall be using the 'moving frames' method. We shall need to be able to do calculations with tangent vectors at each point on the route of a curve in $\mathbf{E}^3$. These first two sections also give a reminder of the link between distances, as usually measured in $\mathbf{E}^3$, and dot products.

In Sections 3 and 4 the study of curves proper begins. First of all a carefully chosen set of basis vectors is placed at each point on the curve. Next, the rates of change of these basis vectors are found and expressed in terms of the basis. The formulas expressing the rates of change of the basis as linear combinations of the basis vectors are of central importance in the study of curves. The formulas are known as the Frenet (sometimes Serret–Frenet) formulas.

Where it seems harmless we shall refer to 'a point on the curve' where we should really refer to 'a point on the route of the curve'.

We shall eventually prove two theorems: the first says, essentially, that congruent curves have the same coefficients in their respective Frenet formulas. The second is the converse: that curves with the same coefficients in their Frenet formulas must be congruent.

We shall formally define congruence later but the formal definition will simply reflect intuition.

These theorems provide a strong link between the calculus of curves and their geometry. The Frenet formulas arise from the calculus by considering the rate of change of a basis. However, they completely determine the geometry of the curve. Perhaps the following, very non-rigorous suggestion may indicate that this is reasonable. Imagine driving along a road at constant speed. The forces that you experience pushing you from side to side (which are related to acceleration—a calculus notion) depend on how sharply curved the road is, compared with the speed with which you drive. Thus links between geometry and calculus are part of everyday experience.

Having shown, in the context of curves, that rates of change of basis vectors can provide geometric information, Sections 5–8 generalize the idea. Instead of just considering bases attached to curves, we consider a set of three vector fields providing an orthonormal basis at each point of $\mathbf{E}^3$. Such a set of vector fields is a frame field. We go on to show how the rate of change of a frame field may be expressed in terms of the frame field itself by using the notion of a 1-form introduced in Part I. Frenet-like formulas appear here, too. These ideas will not be fully exploited until we come to study surfaces.

Section 9 provides the usual end-of-chapter summary.

Study advice

The following represents a possible plan for study weeks.

Week 1 *O'Neill*, Chapter II, Sections 1–2 and TMA 01.

Week 2 *O'Neill*, Chapter II, Sections 3–5.

Week 3 *O'Neill*, Chapter II, Sections 6–8.

This plan leaves two study weeks for the sections of Part III required for TMA02 and for doing the TMA.

Sections 5–8 contain more general ideas than do Sections 1–4, and you may find them rather harder going.

1 Dot product

Read *O'Neill: Chapter II, Section 1, pages 42–48.*

You have probably already used dot and cross products of vectors to discuss orthogonality. In this section we formally transfer these ideas from vectors to tangent vectors.

The section begins with a review of these products for vectors. The properties on page 43 are probably familiar, even if you have not consciously thought about them.

Norm You may have used the notation $|\mathbf{p}|$ instead of $||\mathbf{p}||$ for norms. The double vertical bars for norms are used in this course, reserving single bars for the modulus function.

The norm of $\mathbf{p}$ is the same as the distance from the origin of the point represented by $\mathbf{p}$.

To calculate the dot or cross product of two tangent vectors at a particular point, we simply do the usual calculations with their vector parts, as Definition 1.3 shows for dot products.

There are alternatives to the approach in *O'Neill*. One such approach assumes that angles can be defined independently and *defines*

$$\mathbf{p} \cdot \mathbf{q} = ||\mathbf{p}||\,||\mathbf{q}|| \cos\theta.$$

From this definition, it is fairly easy to show that

$$\mathbf{p} \cdot \mathbf{q} = p_1q_1 + p_2q_2 + p_3q_3.$$

The approach in *O'Neill* is the more usual one and $\mathbf{p} \cdot \mathbf{q}$ is used to define the notion of angle.

Definition 1.4 This is very important. We shall be studying curves by placing a frame at each point on the curve.

Anticipating Section 5 a little, the vector fields that place a frame at each point form a frame field.

Orthonormal expansion The second part of Theorem 1.5 shows why we concentrate on frames, that is an orthonormal basis at each point. It is usually an effort (involving the solution of simultaneous equations) to find the coefficients in the expression of a vector as a linear combination of given basis vectors. For an orthonormal basis, the expression

$$\mathbf{v} = \sum_{i=1}^{3} (\mathbf{v} \cdot e_i)e_i$$

shows that the coefficients are easy to find. This process of orthonormal expansion will be used repeatedly both for theory and calculations.

The language used for frames and orthogonal matrices needs to be noted carefully. If a *basis* is described as orthogonal then all you know is that the vectors are mutually perpendicular. If the basis consists of mutually perpendicular *unit* vectors, then it is described as *orthonormal*.

However, the term orthogonal, when applied to a *matrix*, means that the rows are *orthonormal*.

As we shall see, the columns are also orthonormal.

Transpose The notation that is used in the course for transpose may well be different from the one that you have used before. If ${}^tA$ is read as 'the transpose of A' you may find it a little easier to use and remember.

You may have used $\overline{A}$ or A^* for the transpose of A.

The result

$${}^tA\,A = I$$

follows from direct matrix multiplication using

$$e_i \cdot e_j = \delta_{ij};$$

you do not need to use a 'standard theorem of linear algebra'. In fact, ${}^tAA = I$ is an alternative *definition* of orthogonality for matrices.

We know, from the above equation, that the inverse of an orthogonal matrix A is also the transpose of A:

$$A^{-1} = {}^tA.$$

Since

$$A^{-1}A = I = AA^{-1},$$

it follows that

$$A\,{}^tA = I.$$

We can rewrite this as

$${}^t({}^tA)({}^tA) = I,$$

which shows that the transpose of A is orthogonal and so the columns of A are orthonormal.

Linearity of cross product Explicitly this means that

$$\mathbf{v} \times (\alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2) = \alpha_1\mathbf{v} \times \mathbf{w}_1 + \alpha_2\mathbf{v} \times \mathbf{w}_2$$

and

$$(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2) \times \mathbf{w} = \alpha_1\mathbf{v}_1 \times \mathbf{w} + \alpha_2\mathbf{v}_2 \times \mathbf{w}.$$

This, like the dot product, is an example of bilinearity.

The cross product gives, as Lemma 1.8 shows, a method of constructing a vector orthogonal to two given vectors. We shall use this very often to obtain the third member of a frame after we have found two members. More generally, having found two independent vectors, the cross product will enable us to construct the third member of a basis (though not necessarily a frame).

The length formula for the cross product gives the sine form:

$$\begin{aligned} \|\mathbf{v} \times \mathbf{w}\|^2 &= (\mathbf{v} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{w}) - (\mathbf{v} \cdot \mathbf{w})^2 \\ &= \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - (\|\mathbf{v}\|\,\|\mathbf{w}\| \cos\theta)^2 \\ &= \|\mathbf{v}\|^2\|\mathbf{w}\|^2(1 - \cos^2\theta) \\ &= \|\mathbf{v}\|^2\|\mathbf{w}\|^2 \sin^2\theta. \end{aligned}$$

You may find it easier to visualize the 'right-hand rule', in the last paragraph of page 48, by thinking of a corkscrew in conjunction with the diagram at the top of the page. If the handle rotates from $\mathbf{v}$ to $\mathbf{w}$ then the corkscrew travels in the direction of $\mathbf{v} \times \mathbf{w}$.

It is *possible* that someone has produced left-handed corkscrews! We have only ever seen right-handed ones and it is this type that we suggest that you imagine.

Exercise 1.1 *O'Neill*, page 48, Exercise 1.

Exercise 1.2 *O'Neill*, page 49, Exercise 3.

Exercise 1.3 *O'Neill*, page 49, Exercise 4 (parts (a) and (d) only). You may use the other results to do part (d).

Exercise 1.4 *O'Neill*, page 49, Exercise 5.

Exercise 1.5 *O'Neill*, Page 49, Exercise 6.

Exercise 1.6 *O'Neill*, page 49, Exercise 7.

[*Solutions on page 25*]

2 Curves

Read *O'Neill: Chapter II, Section 2, pages 51–55.*

Erratum *O'Neill*, page 55, the displayed expression for α'' should read

$$\alpha'' = \left(\frac{d^2\alpha_1}{dt^2}, \frac{d^2\alpha_2}{dt^2}, \frac{d^2\alpha_3}{dt^2}\right).$$

This last expression is the same as $(\alpha_1'', \alpha_2'', \alpha_3'')$.

There are two main ideas introduced in this section. The first is that any curve may be reparametrized so as to have unit speed. The second is that it is useful to consider vector fields that are defined only for points on the (route of the) curve.

The proof that a unit-speed reparametrization of a curve can always be found is constructive: the proof provides a method for finding the reparametrization. Although we do ask that you are able to carry out this process in simple cases, we shall make most use of the *existence* of a unit-speed reparametrization rather than the actual reparametrization.

The idea of a vector field defined only for points on a curve is quite natural if you consider the idea of the velocity vector of a curve. The velocity vector $\alpha'(t)$ is a tangent vector at $\alpha(t)$. Thus

$$t \longmapsto \alpha'(t)_{\alpha(t)}$$

defines a tangent vector at each point on the curve, but nowhere else. This is the motivation behind the definition of 'vector field on a curve'.

Proof of Theorem 2.1 Since the derivative of $s(t)$ is positive, s is increasing on its domain and so is one–one. Hence s has an inverse.

Note that this proof provides a technique for finding a unit-speed reparametrization of a curve α. The process can be broken down into stages as follows.

(a) Find the velocity, α', of the curve.

(b) Find the speed, $v(t) = ||\alpha'(t)||$.

(c) Using a suitable 'base point', $\alpha(a)$, find the arc-length function

$$s(t) = \int_a^t v(u)\,du.$$

(d) Invert s to obtain $t = t(s)$.

(e) Find the unit-speed reparametrization $\beta(s) = \alpha(t(s))$.

Parallel vector fields Note that *O'Neill* slips in a definition of parallel vector field. It is defined only for vector fields on a curve but generalizes to the following.

Definition

A vector field on $\mathbf{E}^3$ is *parallel* if the vector part of $V(\mathbf{p})$ is constant.

Straight lines There is a consequence of the proof of Lemma 2.3: the definition of 'straight line' given by *O'Neill* is extremely restrictive. A curve is a straight line if, and only if, its velocity is non-zero and its acceleration is zero. Thus, many curves whose routes are straight lines, in the usual sense, are excluded from this definition. For example: if $\mathbf{p}$ and $\mathbf{v}$ are vectors in $\mathbf{E}^3$, with $\mathbf{v}$ non-zero, then

$$\alpha(t) = \mathbf{p} + t\mathbf{v}, \quad t \in \mathbf{R}$$

defines a straight line, whereas

$$\alpha(t) = \mathbf{p} + t^2\mathbf{v}, \quad t \in \mathbf{R}$$

does not because the acceleration is $2\mathbf{v}$ and is non-zero. However, both have the same route.

Exercise 2.1 This question concerns the curve defined by

$$\alpha(t) = (t\cos t, t\sin t, t), \quad t \in \mathbf{R}.$$

(a) Find the velocity, speed and acceleration of α.

(b) Show that the route of α passes through the origin and find the velocity, speed and acceleration at the origin.

Exercise 2.2 *O'Neill*, page 55, Exercise 3.

Exercise 2.3 *O'Neill*, page 55, Exercise 4.

Exercise 2.4 *O'Neill*, page 56, Exercise 8.

[*Solutions on page 26*]

3 The Frenet formulas

Read *O'Neill: Chapter II, Section 3, pages 56–63.*

This section is the first application of the 'method of moving frames'. We define a frame at each point on a curve and express the derivatives of the resulting vector fields in terms of the fields themselves.

Since we are dealing here with unit-speed curves, the velocity vector (field) provides a ready made unit vector field on the curve. The major step in getting useful information about curves is in defining the other two vector fields appropriately.

The work in this section is in three parts.

(a) The definitions of the vector fields and functions associated with a curve (the **Frenet apparatus** of the curve).

(b) The relationships between the derivatives of the vector fields and the fields themselves (the **Frenet formulas**).

(c) Examples of the use of the Frenet formulas.

The Frenet apparatus and formulas The treatment in *O'Neill* has the definitions, results and proofs interwoven. There follows a summary of the essential points with the definitions, results and proofs separated. You may find this summary rather easier to use later on, for revision.

This interweaving is reasonable from the teaching point of view but may make revision harder, hence our summary.

Definitions For a unit-speed curve

$$\beta : I \longrightarrow \mathbf{R}, \qquad I \text{ an interval in } \mathbf{R},$$

with non-vanishing acceleration, we define the Frenet apparatus of β to be T, κ, N, B and τ, where

$T = \beta'$ is the unit tangent vector field,

$\kappa = \|T'\|$ is the curvature,

$N = \dfrac{T'}{\kappa}$ is the unit normal vector field,

$B = T \times N$ is the unit binormal vector field,

$\tau = -B' \cdot N$ is the torsion.

Note: The definition of τ differs slightly from that in *O'Neill*; this is because the definition in *O'Neill* is inextricably bound up with the proofs of the formulas. Separating the definitions out requires the above. You will find that *O'Neill* makes reference to this definition of τ a little later on.

Frenet formulas With the assumptions and definitions as above

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N. \end{aligned}$$

Note: You may care to remember these formulas in the following 'matrix' form.

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

This form has the advantage of clearly showing the patterns in the coefficients; it also has strong similarities to a generalization of the Frenet formulas that will be discussed in Section 7.

Proofs If you re-read the proofs in *O'Neill* carefully, you will see that they hinge on repeated use of two ideas: orthonormal expansion and the Leibniz property of differentiation. Specifically, we shall want to express a vector field, Y say, on β as

$$Y = (Y \cdot T)T + (Y \cdot N)N + (Y \cdot B)B.$$

This uses orthonormal expansion.

We shall also want to use

$$(Y \cdot Z)' = Y' \cdot Z + Y \cdot Z'$$

This is the Leibniz property.

for vector fields Y and Z on β.

Before we can use orthonormal expansion, we must be sure that T, N and B are orthonormal at each point. To do so we use the definitions of the Frenet apparatus. This is essentially the content of Lemma 3.1. We gather the scattered elements of the proof together here.

Proof of Lemma 3.1 Since β is unit speed, we have

$$||T|| = ||\beta'|| = 1.$$

Consider the consequences of the definitions of N and κ:

$$\begin{aligned} ||N|| &= \frac{||T'||}{\kappa} \qquad \text{(definition of } N\text{)} \\ &= \frac{||T'||}{||T'||} \qquad \text{(definition of } \kappa\text{)} \\ &= 1. \end{aligned}$$

Because $T \cdot T = 1$, we have

$$\begin{aligned} & T' \cdot T + T \cdot T' = 0 \quad \text{(Leibniz)} \\ & \Rightarrow 2T \cdot T' = 0 \\ & \Rightarrow T \cdot T' = 0. \end{aligned}$$

But now it follows that

$$T \cdot N = T \cdot \frac{T'}{\kappa} = 0.$$

Thus T and N are unit length and orthogonal at each point of β.

Finally,

$$B = T \times N$$

is, by definition of cross product, orthogonal to T and N. Also, since T and N are orthogonal,

$$||B|| = ||T||\,||N|| \sin\frac{\pi}{2} = 1.$$

This completes the proof that T, N and B form an orthonormal basis at each point of β.

Note how we needed to know that T and N were *orthogonal* unit vectors before we could deduce that

$$B = T \times N$$

was a unit vector.

With Lemma 3.1 and the definitions of the Frenet apparatus, we can actually prove all three of the Frenet formulas using orthonormal expansion. If we write out the orthonormal expansions of T', N' and B', we obtain

$$\begin{aligned}
T' &= (T' \cdot T)T + (T' \cdot N)N + (T' \cdot B)B \\
N' &= (N' \cdot T)T + (N' \cdot N)N + (N' \cdot B)B \\
B' &= (B' \cdot T)T + (B' \cdot N)N + (B' \cdot B)B.
\end{aligned}$$

The definition

$$N = \frac{T'}{\kappa}$$

shows that the first expansion reduces to

$$T' = \kappa N,$$

and so we can deduce that

$$T' \cdot T = 0$$

(which we have shown independently) and that

$$T' \cdot B = 0.$$

To prove the other two formulas, we must calculate the coefficients; it is here that we again use the Leibniz property.

We have already differentiated $T \cdot T = 1$ to show that

$$T' \cdot T = 0.$$

A similar argument shows that

$$N' \cdot N = 0 \quad \text{and} \quad B' \cdot B = 0.$$

Next we tackle $N' \cdot T$. We have

$$\begin{aligned}
&N \cdot T = 0 \\
&\Rightarrow N' \cdot T + N \cdot T' = 0 \\
&\Rightarrow N' \cdot T = -N \cdot T' \\
&\Rightarrow N' \cdot T = -N \cdot \kappa N = -\kappa N \cdot N = -\kappa.
\end{aligned}$$

A similar argument, starting from $N \cdot B = 0$, shows that

$$N' \cdot B = -B' \cdot N = -(-\tau) = \tau \quad \text{(using the definition of } \tau\text{)}.$$

Finally, starting from $B \cdot T = 0$ gives

$$B' \cdot T = -B \cdot T' = -0 = 0.$$

Assembling all this information gives

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

If we denote the above matrix of coefficients by

$$F = (f_{ij}), \quad i, j = 1, 2, 3,$$

then the entries f_{ij} have the property

$$f_{ji} = -f_{ij}, \quad i, j = 1, 2, 3.$$

We can express this in terms of the transpose as

$$^tF = -F.$$

This 'skew-symmetry' arises from one property of the geometry and one of the calculus. The geometric property is that T, N and B are a frame (orthonormal basis) at each point of β.

The Leibniz property is then applied to

$$Y \cdot Z = \text{constant}$$

to give

$$Y' \cdot Z = -Y \cdot Z'$$

with Y and Z replaced by all possible choices of T, N and B.

It may well occur to you that any similar situation involving derivatives of frames will give rise to similar skew-symmetry amongst the coefficients. You would be correct; we shall consider one other such case in Section 7.

Finding Frenet apparatus The definitions of the Frenet apparatus are constructive. That is, they can be used to find the apparatus of a given unit-speed curve with non-vanishing acceleration. *O'Neill* has a worked example in Example 3.3 (pages 58–59). In this example τ could as easily be calculated from

$$\tau = -B' \cdot N$$

as by comparison of B' and N.

Curves in $\mathbf{E}^2$ We have set you one exercise on curves in $\mathbf{E}^2$. The situation for curves in $\mathbf{E}^2$ is somewhat simpler than for curves in $\mathbf{E}^3$ and the definition of κ is a little different. We shall not consider such curves further, although we shall consider curves in $\mathbf{E}^3$ which happen to lie in a plane.

We know, from *O'Neill*, Corollary 3.5, that $\tau = 0$ for a plane curve.

Interpretation of Frenet apparatus The vector (field) T on a curve is, by definition, bound up with the geometric notion of 'tangent'. The other items in the Frenet apparatus also have geometric interpretations.

The normal, N, is in the direction of the rate of change of the tangent, by virtue of

$$T' = \kappa N.$$

Firstly, this shows that the tangent is 'turning' towards N. If we were to approximate the curve by a circle near some point, then N would point towards the centre of the circle, that is, towards the concave side of the curve.

Secondly, this formula also shows that κ measures the magnitude of the rate of turning of the tangent. The rate at which the tangent turns is, intuitively, a sensible way of measuring 'curvature'. This is why κ is defined to be the curvature function.

What is not so obvious, but can be shown in various ways, is that the 'best approximation' to a curve near $\alpha(t)$ by a *circle* is the circle which has the same tangent to the curve at the point, lies in the T–N plane and has radius $1/\kappa$.

Intuitively, if you want a plane approximation to the curve near some point, then the T–N plane is where you will have to look. Since B is, by definition, perpendicular to this plane, the rate of change of B will measure how the plane approximation varies from point to point. It is not surprising, therefore, that Corollary 3.5 shows that $\tau = 0$, which is equivalent to $B' = 0$, is the correct condition for detecting curves which always lie in a plane.

Exercise 3.1 *O'Neill*, page 63, Exercise 1.

Exercise 3.2 *O'Neill*, page 63, Exercise 2.

Exercise 3.3 *O'Neill*, page 64, Exercise 5.

Exercise 3.4 *O'Neill*, page 64, Exercise 6.

Exercise 3.5 *O'Neill*, page 65, Exercise 8. (See comment above about curves in the plane.)

[*Solutions on page 27*]

4 Arbitrary-speed curves

Read *O'Neill: Chapter II, Section 4, pages 66–74.*

This section extends the Frenet formulas to arbitrary speed curves. This is an important extension for practical computational reasons. Although a unit-speed reparametrization always *exists* (for the curves that we consider), actually finding it is often impractical.

The work in this section falls into three parts.

(a) The definition of the Frenet apparatus for (regular) curves in general and the consequent versions of the Frenet formulas.

(b) Obtaining practical computational formulas for the Frenet apparatus of such curves.

(c) Examples of the use of the formulas.

General Frenet apparatus The definitions say very much what one might expect: at any point the Frenet apparatus is defined to be the Frenet apparatus of the unit-speed reparametrization at the corresponding point. It is common to include the speed v when discussing the Frenet apparatus.

General Frenet formulas Equally, the appearance of the speed

$$s'(t) = v(t)$$

is predictable, given the central role that the chain rule plays when considering the effect of 'change of variable' on derivatives. The 'correction factor' that appears when changing variable from s to t is precisely $s'(t) = v(t)$. If we have a function F defined on the curve, then

$$\frac{dF}{dt} = \frac{dF}{ds}s'(t) = \frac{dF}{ds}v(t).$$

Once again, you may find the formulas more memorable in matrix form.

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa v & 0 \\ -\kappa v & 0 & \tau v \\ 0 & -\tau v & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

In theory, the general Frenet formulas would enable the Frenet apparatus of a curve such as

$$\alpha(t) = (t, t^2, t^3), \quad t \in \mathbf{R}$$

to be calculated. However, the calculations would still be very messy, as the following might indicate.

It would not be possible to find the unit-speed parametrization for this curve in explicit form, as you might like to check by experiment. The resulting integral for s cannot be expressed in terms of 'elementary functions'.

First, we could find the velocity,

$$\alpha'(t) = (1, 2t, 3t^2).$$

From this we get

$$v(t) = \sqrt{1 + 4t^2 + 9t^4}.$$

Hence, using $\alpha' = vT$,

$$T = \frac{1}{\sqrt{1 + 4t^2 + 9t^4}}(1, 2t, 3t^2).$$

Differentiating this expression for T in order to find N from

$$T' = \kappa v N$$

is rather unpleasant!

The next step in *O'Neill* is to provide formulas for the Frenet apparatus in terms of α and its first three derivatives.

Computational results The formulas provided by Theorem 4.3 get round the problem indicated above.

The method of proof is typical of all uses of the Frenet formulas: differentiate and then use the Frenet formulas to replace derivatives of the frame vectors by combinations of the vectors themselves. Note that the differentiations usually involve use of the Leibniz property in one or other of its manifestations.

Special curves At this point *O'Neill* introduces two special classes of curves: spherical images and cylindrical helices.

Note that we adopt the 'classic' plural of helix: helices, in spite of using the non-classical plural 'formulas'!

The spherical image of a curve is defined by ignoring the point of application of T 'transferring T to the origin' as *O'Neill* puts it. Thus, the coordinate functions of T are used to define the spherical image curve.

You have already met the circular helix which is a special case of the cylindrical helix. Note that, as Fig. 2.19 shows, the 'cylinder' in 'cylindrical helix' should *not* be taken to imply 'circular cylinder'. The cross-section curve can be any plane curve.

Exercise 4.1 *O'Neill*, page 74, Exercise 1(a).

Exercise 4.2 *O'Neill*, page 74, Exercise 2.

Exercise 4.3 *O'Neill*, page 74, Exercise 3(a).

Exercise 4.4 *O'Neill*, page 74, Exercise 6. (*Hint*: Use the Frenet formulas and the fact that the speed is constant to express the derivatives of α in terms of the Frenet apparatus. Then adapt the proof of Lemma 4.2.)

Exercise 4.5 *O'Neill*, pages 74–5, Exercise 8.

Exercise 4.6 *O'Neill*, page 75, Exercise 10. (*Hint*: A circular helix is a cylindrical helix whose cross-section curve is a circle.)

Exercise 4.7 *O'Neill*, page 75, Exercise 11.

Note: Use the results obtained in *O'Neill* for σ', σ'' and κ_σ and just calculate τ_σ.

Exercise 4.8 *O'Neill*, page 76, Exercise 12(a).

[*Solutions on page 29*]

5 Covariant derivatives

Read *O'Neill: Chapter II, Section 5, pages 77–80.*

Errata

1 On page 80, Exercise 1, the second line:

for 'Compute $\nabla_{\mathbf{v}} \ldots$' read 'Compute $\nabla_{\mathbf{v}_{\mathbf{p}}} \ldots$'

2 On page 80, Exercise 5, the second sentence would probably read better as

'Thus ∇W is the ...'. ■

We now pause in our study of curves to begin the generalization of the method of moving frames. With a curve, the three vector fields T, N and B that provide the Frenet frame are functions of the single variable used to define the curve. The derivatives of the Frenet frame are, therefore, just the ordinary derivatives of elementary calculus. On a curve there is only one way to go: along the curve.

This generalization will take up Sections 5–8, we shall return to curves in Part III.

In general, there will be the possibility of moving in various directions from a given point so we need the concept of a directional derivative for vector fields. The idea of **covariant derivative** provides what we need. It is the direct analogue for vector fields of the directional derivative for functions from $\mathbf{E}^3$ to $\mathbf{R}$.

The covariant derivative $\nabla_{\mathbf{v}_{\mathbf{p}}} W$ measures the initial rate of change of the value of W, as you set off from $\mathbf{p}$ with velocity $\mathbf{v}$. Because values of W are tangent vectors, $\nabla_{\mathbf{v}_{\mathbf{p}}} W$ is a tangent vector based at $\mathbf{p}$.

Lemma 5.2 This is useful as it relates the new form of derivative to one that you have already met: the directional derivative of a function. Paraphrasing: to differentiate a vector field, differentiate the coordinate functions; exactly what you are used to doing with the Frenet vector fields.

Theorem 5.3 You should be expecting this sort of result to appear after each definition of a new form of derivative: it states that the new derivative has appropriate linearity and Leibniz properties. Parts (3) and (4) are not, perhaps, what you might write down immediately if asked to produce Leibniz properties for $\nabla_{\mathbf{v}_{\mathbf{p}}} W$. However, they are actually inevitable!

For example, consider part (3). To differentiate the product fY with respect to $\mathbf{v}_{\mathbf{p}}$, we have to differentiate each of f and Y with respect to $\mathbf{v}_{\mathbf{p}}$. Since f is a function, its derivative will be $\mathbf{v}_{\mathbf{p}}[f]$. The derivative of Y will be $\nabla_{\mathbf{v}_{\mathbf{p}}} W$. Finally, if we remember that the end result must be a tangent vector based at $\mathbf{p}$, we know that Y in the first term and f in the second must be evaluated at $\mathbf{p}$.

To understand that last point, consider a 'first guess':

$$\mathbf{v}_{\mathbf{p}}[f]\, Y + f \nabla_{\mathbf{v}_{\mathbf{p}}} Y.$$

Warning: this first guess is **incorrect!**

This is entirely reasonable *except* that it gives a result of the wrong sort. Actually, the first term is a vector field and the second is undefined (being a function multiplied by a tangent vector).

Examples It is instructive to consider three special cases of covariant derivatives, those with respect to $U_1(\mathbf{p})$ etc.

First we calculate

$$\nabla_{(1,0,0)_{\mathbf{p}}} W,$$

for an arbitrary vector field W with coordinate functions w_1, w_2 and w_3.

Applying Lemma 5.2 gives

$$\nabla_{(1,0,0)_{\mathbf{p}}} W = (1,0,0)_{\mathbf{p}}[w_1]U_1(\mathbf{p}) + (1,0,0)_{\mathbf{p}}[w_2]U_2(\mathbf{p}) + (1,0,0)_{\mathbf{p}}[w_3]U_3(\mathbf{p}).$$

However, we know from Chapter I that

$$(1,0,0)_{\mathbf{p}}[w_i] = \frac{\partial w_i}{\partial x}(\mathbf{p}).$$

Thus,

$$\nabla_{(1,0,0)_{\mathbf{p}}} W = \left(\frac{\partial w_1}{\partial x}(\mathbf{p}), \frac{\partial w_2}{\partial x}(\mathbf{p}), \frac{\partial w_3}{\partial x}(\mathbf{p})\right)_{\mathbf{p}}.$$

Paraphrasing: to differentiate with respect to $U_1(\mathbf{p})$, partially differentiate with respect to x (and evaluate at $\mathbf{p}$).

It seems entirely reasonable that directional differentiation (with unit speed) in the x-direction should reduce to $\partial/\partial x$.

Similar calculations show that

$$\nabla_{(0,1,0)_{\mathbf{p}}} W = \left(\frac{\partial w_1}{\partial y}(\mathbf{p}), \frac{\partial w_2}{\partial y}(\mathbf{p}), \frac{\partial w_3}{\partial y}(\mathbf{p})\right)_{\mathbf{p}},$$

$$\nabla_{(0,0,1)_{\mathbf{p}}} W = \left(\frac{\partial w_1}{\partial z}(\mathbf{p}), \frac{\partial w_2}{\partial z}(\mathbf{p}), \frac{\partial w_3}{\partial z}(\mathbf{p})\right)_{\mathbf{p}}.$$

You might like to note that these results are entirely consistent with the results

$$(1,0,0)_{\mathbf{p}}[f] = \frac{\partial f}{\partial x}(\mathbf{p})$$

etc., that we obtained earlier.

Extension to vector fields *O'Neill* follows the usual pattern next by extending the definition of covariant derivative to allow differentiation with respect to a vector field. The method of extension is exactly the same as for directional derivatives.

The definition is essentially that of a composite function:

$$\nabla_V W : \mathbf{p} \longmapsto V(\mathbf{p}) \longmapsto \nabla_{V(\mathbf{p})} W.$$

All definitions based on the pointwise principle are, effectively, composite function definitions.

At each point, differentiate W with respect to the tangent vector provided by V. This gives a function producing a tangent vector at each point, that is a vector field. Thus the covariant derivative of a vector field *with respect to a vector field* is also a vector field.

Because of this last remark, the results in Corollary 5.4 are exactly what a 'first guess' might well produce. (Compare the remarks about Theorem 5.3 made above.)

Examples The three special cases discussed above yield much less involved formulas when extended to differentiation with respect to the natural frame field.

For example, $\nabla_{U_1} W$ is, by definition,

$$\nabla_{U_1} W : \mathbf{p} \longmapsto U_1(\mathbf{p}) \longmapsto \nabla_{(1,0,0)_{\mathbf{p}}} W.$$

Using what we obtained above gives

$$\nabla_{U_1} W = \frac{\partial w_1}{\partial x} U_1 + \frac{\partial w_2}{\partial x} U_2 + \frac{\partial w_3}{\partial x} U_3.$$

In other words: to differentiate with respect to U_1, partially differentiate with respect to x.

Similarly,

$$\nabla_{U_2} W = \frac{\partial w_1}{\partial y} U_1 + \frac{\partial w_2}{\partial y} U_2 + \frac{\partial w_3}{\partial y} U_3$$

$$\nabla_{U_3} W = \frac{\partial w_1}{\partial z} U_1 + \frac{\partial w_2}{\partial z} U_2 + \frac{\partial w_3}{\partial z} U_3.$$

Note that these results correspond exactly to $U_1[f] = \partial f/\partial x$ etc.

Calculation techniques To calculate a given covariant derivative, repeated application of linearity will reduce the problem to derivatives with respect to the natural frame field. These derivatives can then be found by straightforward partial differentiation.

An extension The definition of the covariant derivative $\nabla_{\mathbf{v}_\mathbf{p}} W$ given in this section is fine when the vector field W is defined on the whole of a region of $\mathbf{E}^3$ surrounding the point $\mathbf{p}$ because we can be sure that $W(\mathbf{p}+t\mathbf{v})$ is actually defined near $\mathbf{p}$. However, we shall want to apply these ideas on surfaces, where the vector fields being differentiated are defined only for points on the surface. The line $\mathbf{p}+t\mathbf{v}$ will probably leave the surface immediately you move away from $\mathbf{p}$, so we need an alternative definition of covariant derivative.

The basis of the alternative definition is contained in Exercise 6, page 81 of *O'Neill*. However, the idea is important enough for us to want to look at it now, rather than leaving it as an exercise.

Suppose that we have a curve

$$\alpha : t \longmapsto \alpha(t), \quad t \in I,$$

and a vector field W that is defined in a region that includes the route of α. Then we can form the composite

$$W(\alpha) : t \longmapsto W(\alpha(t)).$$

This composite has an ordinary derivative at t, namely $(W(\alpha))'(t)$, which is a tangent vector at $\alpha(t)$.

We can also covariantly differentiate W with respect to the tangent $\alpha'(t)$ at $\alpha(t)$. This will also be a tangent vector based at $\alpha(t)$. It is reasonable to ask how these two derivatives are related. The answer is that they are exactly the same:

$$\nabla_{\alpha'(t)} W = (W(\alpha(t)))'.$$

The proof involves applying *O'Neill*, Chapter I, Lemma 4.6 three times, once to each of the coordinate functions of W.

This lemma is the corresponding result to *O'Neill*, Chapter I, 4.6 for directional derivatives of functions.

It is less important to understand all the details of the proof than to appreciate the result and its consequences. Suppose that we have a curve α with the following properties:

$$\alpha(0) = \mathbf{p}, \quad \alpha'(0) = \mathbf{v}.$$

Now apply the above result at $t = 0$:

$$\begin{aligned}\nabla_{\mathbf{v}_\mathbf{p}} W &= \nabla_{\alpha'(0)} W \\ &= (W(\alpha(t)))'(0).\end{aligned}$$

Let us look at what this says a little more carefully. If we define

$$\beta(t) = \mathbf{p} + t\mathbf{v},$$

then the definition of covariant derivative becomes

$$\nabla_{\mathbf{v}_\mathbf{p}} W = (W(\beta(t)))'(0).$$

Now, it is fairly clear that

$$\beta(0) = \mathbf{p} \quad \text{and} \quad \beta'(0) = \mathbf{v}.$$

What we now know is that the straight line β can be replaced by *any* curve which goes through the point $\mathbf{p}$ with velocity $\mathbf{v}$, that is any curve α such that

$$\alpha(0) = \mathbf{p}, \quad \alpha'(0) = \mathbf{v}.$$

When we come to deal with surfaces, this result will become the *definition* of covariant derivative.

Exercise 5.1 *O'Neill*, page 80, Exercise 1.

Exercise 5.2 *O'Neill*, page 80, Exercise 2 (parts (a), (c) and (e) only).

Exercise 5.3 *O'Neill*, page 80, Exercise 4.

Exercise 5.4 *O'Neill*, page 80, Exercise 5. Errata

Exercise 5.5 Let the curve α and the vector field W be defined by

$$\alpha(t) = (1 + \sin t, 3 - \sin t, t^2 + 2t - 1),$$
$$W = x^2 U_1 + y U_2.$$

Show that

$$\alpha(0) = (1, 3, -1) \quad \text{and} \quad \alpha'(0) = (1, -1, 2).$$

Calculate

$$W(\alpha(t))$$

and hence find

$$\nabla_{(1,-1,2)_{(1,3,-1)}} W.$$

Compare your answer with the one you gave for the first exercise in this set.

[*Solutions on page 32*]

6 *Frame fields*

Read *O'Neill: Chapter II, Section 6, pages 81–84.*

In the last section we discussed the notion of derivative, the covariant derivative, that we require for generalizing the Frenet formulas. In this section we formally define a **frame field**.

The definition was anticipated when we referred to U_1, U_2 and U_3 as the natural *frame field*. The U_i provide an orthonormal basis at each point of $\mathbf{E}^3$.

The basis that *M203* refers to as the standard basis.

The preamble to Definition 6.1 and the definition itself formalize the idea of a set of vector fields providing an orthonormal basis at each point.

Examples We strongly recommend that you construct your own versions of Figures 2.21–2.23 of *O'Neill*; it is always difficult to follow someone else's sketches of three-dimensional arrangements.

Lemma 6.3 What *O'Neill* means by 'immediate consequence' is the following.

If $\mathbf{p} \in \mathbf{E}^3$, then $V(\mathbf{p})$ can be expressed in terms of the $E_i(\mathbf{p})$ by orthonormal expansion:

$$V(\mathbf{p}) = (V(\mathbf{p}) \cdot E_1(\mathbf{p}))E_1(\mathbf{p}) + (V(\mathbf{p}) \cdot E_2(\mathbf{p}))E_2(\mathbf{p}) + (V(\mathbf{p}) \cdot E_3(\mathbf{p}))E_3(\mathbf{p}).$$

By invoking the pointwise principle, we obtain the 'functional' form of the above:

$$V = (V \cdot E_1)E_1 + (V \cdot E_2)E_2 + (V \cdot E_3)E_3.$$

The expressions $V \cdot E_i$ define the coordinate functions f_i of V with respect to the frame field E_i.

Exercise 6.1 *O'Neill*, page 84, Exercise 1.

Note: This result is, in theory, quite useful. It gives a method of starting with two vector fields that are merely linearly independent and converting them into orthogonal unit vector fields. (The third member of a frame is then generated using the cross product.) In practice we shall use this technique only on one occasion, in connection with surfaces.

Exercise 6.2 *O'Neill*, page 84, Exercise 2, for the cylindrical frame field only.

[*Solutions on page 32*]

7 Connection forms

Read *O'Neill: Chapter II, Section 7, pages 85–90.*

We now come to the main idea of Sections 5–8: expressing the derivatives of an arbitrary frame field in terms of the frame field itself. We want an analogue of the Frenet formulas:

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

We are looking, informally, for something that looks like

$$(\text{derivative of}) \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = (\text{suitable matrix}) \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}.$$

We know what sort of derivative we must use: the covariant derivative. Thus we aim to express the three covariant derivatives

$$\begin{pmatrix} \nabla_{\mathbf{v_p}} E_1 \\ \nabla_{\mathbf{v_p}} E_2 \\ \nabla_{\mathbf{v_p}} E_3 \end{pmatrix}$$

in terms of $E_1(\mathbf{p})$, $E_2(\mathbf{p})$, $E_3(\mathbf{p})$.

The preamble to Lemma 7.1 has these ideas written out in full. In matrix form the equations become

$$\begin{pmatrix} \nabla_{\mathbf{v_p}} E_1 \\ \nabla_{\mathbf{v_p}} E_2 \\ \nabla_{\mathbf{v_p}} E_3 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} E_1(\mathbf{p}) \\ E_2(\mathbf{p}) \\ E_3(\mathbf{p}) \end{pmatrix}.$$

In any particular case the coefficients c_{ij} can be calculated by the usual methods of orthonormal expansion. However, we shall develop a general method of calculating them from information about the frame field E_i, $i = 1, 2, 3$.

Lemma 7.1 Since the coefficients c_{ij} above are real numbers which depend on the tangent vector being used to differentiate the frame field, it is inevitable that the functions

$$\mathbf{v_p} \longmapsto c_{ij}(\mathbf{v_p})$$

will turn out to be 1-forms. Indeed, anticipation of this result was one of the main reasons for introducing 1-forms.

The proof of the lemma uses only the linearity properties of covariant differentiation.

The other main property of covariant differentiation, the Leibniz property, also has consequences for the coefficients. Just as with the Frenet formulas, the matrix of coefficients is skew-symmetric.

We can summarize the results of Lemma 7.1 and Theorem 7.2 in matrix form as

$$\nabla_V \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} (V) \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

in which form the analogy with the Frenet formulas is fairly clear.

Note that, just as with the Frenet formulas, the proofs used orthonormal expansion, the frame properties and the Leibniz property of differentiation. Not only do the results look similar, so do the methods of obtaining them. The Leibniz property gives rise to the skew-symmetry and, hence, to the zeros down the main diagonal. The other zeros that appeared in the Frenet matrix were due to the very special way in which the Frenet frame T, N, B was chosen.

Calculating connection forms In any particular example of frame fields, the frame will often be given in terms of the natural frame field. *O'Neill* now shows how to calculate the connection forms directly, if you are given the coefficients expressing a frame field E_i, $i = 1, 2, 3$, in terms of the natural frame field.

The proof of Theorem 7.3 applies the definition of covariant derivative and some of the lemmas proved earlier. The method of proof is interesting but much less important than the result. Once again, the matrix form is probably the more memorable:

$$\omega = dA\,{}^tA.$$

The Frenet formulas Rather than leave you to deduce the Frenet formulas from the connection equations (page 91, Exercise 8 of *O'Neill*), we present here one way of doing the deduction.

Suppose that we have a unit-speed curve

$$\alpha : s \longmapsto \alpha(s),$$

and a frame field whose restriction to the curve gives the Frenet frame. That is, specifically, E_1, E_2 and E_3 is a frame field and

The frame field is defined on enough of $\mathbf{E}^3$ to include all of the route of α.

$$E_1(\alpha) = T,$$
$$E_2(\alpha) = N,$$
$$E_3(\alpha) = B.$$

Now we need the link between differentiation with respect to s and covariant derivatives. The link is provided by the result we proved earlier:

This is differentiation 'along the curve'.

$$\nabla_{\alpha'} V = (V(\alpha))'.$$

We have

$$\begin{aligned} T'(s) &= (E_1(\alpha(s)))' \\ &= \nabla_{\alpha'} E_1. \end{aligned}$$

But we know that

$$\alpha' = T,$$

so, doing similar calculations for N' and B', we have

$$T' = \nabla_T T,$$
$$N' = \nabla_T N,$$
$$B' = \nabla_T B.$$

Now we can compare the results given by the definitions of the Frenet apparatus with the expressions for the covariant derivatives from the connection equations.

From the connection equations,

$$\nabla_T T = \omega_{11}(T)\,T + \omega_{12}(T)\,N + \omega_{13}(T)\,B.$$

We know that $\omega_{11} = 0$ for any frame field and the definitions of κ and N imply that

$$T' = \kappa N.$$

Thus

$$\omega_{12}(T) = \kappa,$$
$$\omega_{13}(T) = 0.$$

Note that the immediate consequences of the skew-symmetry of the connection equations are that

$$\omega_{31}(T) = -\omega_{13}(T) = 0,$$
$$\omega_{21}(T) = -\omega_{12}(T) = -\kappa.$$

The remaining independent connection form is ω_{23}. We can find this by considering B' and the definition of τ as $-B' \cdot N$:

$$\begin{aligned} B' &= \nabla_T B \\ &= \omega_{31}(T)\,T + \omega_{32}(T)\,N + \omega_{33}(T)\,B \\ &= 0 - \tau N + 0. \end{aligned}$$

Thus

$$\omega_{31}(T) = 0, \quad \omega_{32}(T) = -\tau.$$

Hence

$$\omega_{23}(T) = -\omega_{32}(T) = -(-\tau) = \tau.$$

Reasonably enough, the connection forms are known only on the curve. Summarizing, we have

$$\omega_{12}(T) = \kappa,\ \omega_{13}(T) = 0,\ \omega_{23}(T) = \tau.$$

Note that there is no point in even asking questions about expressions such as $\nabla_N T$ etc. This is because moving in the N-direction from a point on the curve immediately takes you off the curve to where T is undefined.

This situation will appear again when we consider surfaces.

Exercise 7.1 *O'Neill*, page 90, Exercise 1.

Exercise 7.2 *O'Neill*, page 90, Exercise 2.

Exercise 7.3 *O'Neill*, page 90, Exercise 3.

Exercise 7.4 *O'Neill*, page 91, Exercise 4.

Exercise 7.5 *O'Neill*, page 91, Exercise 5. (*Hint*: Try applying the appropriate version of the Leibniz property to each of the terms $f_i E_i$.)

[*Solutions on page 33*]

8 The structural equations

Read *O'Neill: Chapter II, Section 8, pages 91–95.*

The work in this section is placed here for the sake of completeness. It will not actually be put to use until well into our study of surfaces. However, logically it belongs with the previous discussion of frame fields.

What we are going to do is to generalize the relationship between the natural frame field U_1, U_2, U_3 and the 1-forms dx, dy, dz.

We know, from *O'Neill*, Chapter I, that the 1-forms dx, dy and dz pick out the coordinates of a tangent vector, that is if

$$\mathbf{v_p} = v_1 U_1(\mathbf{p}) + v_2 U_2(\mathbf{p}) + v_3 U_3(\mathbf{p}),$$

then

$$dx(\mathbf{v_p}) = v_1, \quad dy(\mathbf{v_p}) = v_2, \quad dz(\mathbf{v_p}) = v_3.$$

Definition 8.1 This is, effectively, the generalization referred to above. Because the $E_i(\mathbf{p})$ form a frame at $\mathbf{p}$, we can use orthonormal expansion to give

$$\mathbf{v_p} = (\mathbf{v_p} \cdot E_1(\mathbf{p}))E_1(\mathbf{p}) + (\mathbf{v_p} \cdot E_2(\mathbf{p}))E_2(\mathbf{p}) + (\mathbf{v_p} \cdot E_3(\mathbf{p}))E_3(\mathbf{p}).$$

Thus

$$\theta_i(\mathbf{v_p}) = \mathbf{v_p} \cdot E_i(\mathbf{p})$$

is the *i*th coordinate of the vector part of $\mathbf{v_p}$. Thus θ_i does indeed pick out the E_i-component of $\mathbf{v_p}$.

Note that *O'Neill* slips quietly into applying the dual 1-forms to vector fields, rather than tangent vectors, without any comment. What he is doing is to apply the pointwise principle to define

$$\theta_i(V) : \mathbf{p} \longmapsto V(\mathbf{p}) \cdot E_i(\mathbf{p}).$$

This can be stated more succinctly as

$$\theta_i(V) = V \cdot E_i.$$

Suppose that we have a vector field on $\mathbf{E}^3$ defined in terms of the E_i by

$$V = f_1E_1 + f_2E_2 + f_3E_3,$$

where the f_i, $i = 1, 2, 3$ are functions on $\mathbf{E}^3$. Then, from the above and the properties of a frame field, we have

$$\begin{aligned}\theta_i(V) &= V \cdot E_i \\ &= (f_1E_1 + f_2E_2 + f_3E_3) \cdot E_i \\ &= f_i.\end{aligned}$$

Thus, just as θ_i picks out the E_i-component of a tangent vector, θ_i picks out the E_i-coordinate function of a vector field.

A consequence of the definition of the dual 1-forms is that

$$\theta_i(E_j) = E_j \cdot E_i = \delta_{ij}.$$

This consequence could, in fact, be taken as the *definition* of the dual 1-forms.

Once we know the effect of a 1-form on a frame field, it is uniquely determined, by linearity (see Lemma 8.2 below). Thus, the dual 1-forms are completely determined by the relationships

$$\theta_i(E_j) = E_j \cdot E_i = \delta_{ij}.$$

This relationship, between a frame field and the dual 1-forms, is the easiest to use if you are asked to check that given 1-forms actually are the duals of a given frame field.

Lemma 8.2 This is the generalization of the result from *O'Neill*, Chapter I, that any 1-form can be expressed uniquely as a linear combination of dx, dy and dz. The lemma also provides a way of calculating the coefficients: evaluate the given 1-form on the frame field members.

Next, *O'Neill* shows how the dual 1-forms for a frame field are related to the 'standard' 1-forms dx, dy and dz. The relationship turns out to be exactly the same as between the frame field and the 'standard' frame field, that is, the natural frame field. The *same* attitude matrix does for both.

If the large number of summation signs confuses you, try writing out some of the expressions in full. For example, in the proof that

$$\begin{pmatrix}\theta_1 \\ \theta_2 \\ \theta_3\end{pmatrix} = A \begin{pmatrix} dx \\ dy \\ dz\end{pmatrix},$$

where A is the attitude matrix for the frame, a crucial step is the following.

$$\begin{aligned}\theta_i(U_j) &= U_j \cdot E_i \\ &= U_j \cdot (a_{i1}U_1 + a_{i2}U_2 + a_{i3}U_3) \\ &= a_{ij}.\end{aligned}$$

We have used the fact that $U_i \cdot U_j = \delta_{ij}$. The only term in the sum that survives is the one containing $U_j \cdot U_j$.

The structural equations The equations discovered by Cartan really are quite remarkable.

The discussion above, and in *O'Neill*, shows that the attitude matrix A describes all the 'static' information about a frame field: the frame field itself and its associated dual 1-forms. What the Cartan equations say is that the connection form matrix, ω, describes the rates of change of *everything* associated with a frame field: the covariant derivatives of the frame field, the exterior derivatives of the dual 1-forms and even the derivative of ω itself!

The attitude matrix gives the new frame and new 1-forms in terms of the 'standard' ones.

We think that it is instructive to consider proofs of the various parts of Theorem 8.3 that make more use of matrix methods. First we consider part (1).

In matrix form we have

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = A \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

and, by applying A^{-1} to both sides,

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = A^{-1} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}.$$

Because A is orthogonal, $A^{-1} = {}^tA$ so

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = {}^tA \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}.$$

Differentiating the original expression for the dual 1-forms,

$$\begin{aligned} \begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} &= dA \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \\ &= dA\,{}^tA \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \\ &= \omega \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}. \end{aligned}$$

With care we can give an even more concise matrix proof of part (2). We start from

$$\omega = dA\,{}^tA.$$

Applying the Leibniz property for exterior derivatives (*O'Neill*, Chapter I) we get

$$\begin{aligned} d\omega &= d(dA\,{}^tA) \\ &= d(dA)\,{}^tA - dA\,d({}^tA) \\ &= 0 - dA\,{}^t(dA). \end{aligned}$$

It is tedious, but not difficult, to show that the property also applies to matrices. You just have to consider the elements of the matrices.

To complete the proof the trick is to insert the identity matrix in the form

$$I = A^{-1}A = {}^tA\,A.$$

This gives

$$\begin{aligned} d\omega &= -(dA)\,({}^tA\,A)\,{}^tdA \\ &= -(dA\,{}^tA)(A\,{}^tdA) \\ &= -\omega\,{}^t(dA\,{}^tA) \\ &= -\omega\,{}^t\omega \\ &= \omega\,\omega. \quad \text{(Skew-symmetry of } \omega\text{, i.e. } {}^t\omega = -\omega.) \end{aligned}$$

Note that ${}^t(AB) = {}^tB\,{}^tA$ for any appropriately sized matrices A and B.

The family resemblance between the results that we have obtained in this section is all the more striking if they are all expressed in matrix form.

Where necessary, multiplication should be interpreted as using wedge products.

$$\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = A \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix},$$

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = A \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix},$$

$$\nabla_V \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \omega(V) \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix},$$

$$d \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \omega \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix},$$

$$d\omega = \omega\,\omega.$$

Because we shall not be using these results for some time, we have not set many exercises on this section.

Since some of the exercises require you to show that two 1-forms are equal, we think that a reminder of how to do so is in order.

Being functions, two 1-forms are equal if, and only if, they have the same domain and the same values on all tangent vectors in their domains. Because 1-forms are linear, all that we need to check is that they agree on a basis at each point. In practice this means showing that they agree on a frame field.

To see why this is sufficient, suppose that ϕ and ψ are 1-forms, that E_i, $i = 1, 2, 3$ is a frame field and

$$\phi(E_i) = \psi(E_i), \quad i = 1, 2, 3.$$

If V is any vector field, then

$$V = v_1E_1 + v_2E_2 + v_3E_3.$$

Then

$$\begin{aligned} \phi(V) &= \phi(v_1E_1 + v_2E_2 + v_3E_3) \\ &= v_1\phi(E_1) + v_2\phi(E_2) + v_3\phi(E_3) \\ &= v_1\psi(E_1) + v_2\psi(E_2) + v_3\psi(E_3) \\ &= \psi(V). \end{aligned}$$

Thus ϕ and ψ have the same value on any vector field and are, therefore, equal as 1-forms.

Exercise 8.1 *O'Neill*, page 95, Exercise 1.

Exercise 8.2 *O'Neill*, page 96, Exercise 4.

Suggestion: The method advocated in *O'Neill* leads to rather long-winded calculations. We suggest that you start from the equation connecting the dual 1-forms and dx, dy and dz. Using $x = r\cos\vartheta$ and $y = r\sin\vartheta$ you can express the standard 1-forms in terms of dr, etc.

[*Solutions on page 34*]

9 Summary

Read *O'Neill: Chapter II, Section 9, page 96.*

The main purpose of this part has been to introduce you to the study of curves and the derivation of the Frenet formulas. However, the method of moving frames is of such general importance that we have also included the generalization of the Frenet approach to obtain the connection forms and Cartan's structural equations.

The most important computational techniques that you should be able to carry out are the calculation of the Frenet apparatus for unit-speed and arbitrary-speed curves.

The connection and structural equations will reappear in our study of surfaces.

For a unit-speed curve, the Frenet apparatus gives a description of the curve using only ideas belonging to the curve itself, rather than the description of the curve by three coordinate functions. The parameter s, the arc-length, belongs to the curve, not to where the curve is situated in $\mathbf{E}^3$. Similarly, the curvature κ and torsion τ are functions of s and, in principle at least, a creature confined to the curve, and having no knowledge of how the curve is situated in $\mathbf{E}^3$, could measure s, κ and τ. Such a creature would have to have perception of more than the one dimension of the curve in order to conceive the vectors N and B. Even so, this knowledge need be 'local' only to its current position on the curve; it need have no perception of the curve as a whole.

Nevertheless, s and the two functions κ and τ do give information about what the curve looks like to an observer who can see the curve embedded in $\mathbf{E}^3$.

In Part III (*O'Neill*, Chapter III) we show that knowledge of κ and τ as functions of s is enough to determine everything about a curve *except* its precise location in $\mathbf{E}^3$.

Solutions to the exercises

Solution 1.1

(a) Applying the definition:

$$\mathbf{p}\cdot\mathbf{q} = 1\times(-1)+2\times 0+(-1)\times 3 = -4.$$

(b) We assume that the tangent vectors are at a point $\mathbf{p}$. Using the formal determinant method:

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} U_1(\mathbf{p}) & U_2(\mathbf{p}) & U_3(\mathbf{p}) \\ 1 & 2 & -1 \\ -1 & 0 & 3 \end{vmatrix}$$

$$= U_1(\mathbf{p})\begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} - U_2(\mathbf{p})\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} + U_3(\mathbf{p})\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix}$$

$$= (6,-2,2)_{\mathbf{p}}.$$

(c) Since $\|\mathbf{v}\| = \sqrt{6}$ and $\|\mathbf{w}\| = \sqrt{10}$, we have

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{6}}(1,2,-1),$$

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \frac{1}{\sqrt{10}}(-1,0,3).$$

(d) Using the result of part (b),

$$\|\mathbf{v}\times\mathbf{w}\| = \sqrt{36+4+4} = \sqrt{44} = 2\sqrt{11}.$$

(e) If the angle is θ, we have

$$\begin{aligned}\cos\theta &= \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} \\ &= \frac{-4}{\sqrt{6}\times\sqrt{10}} \\ &= -\frac{4}{\sqrt{2}\sqrt{3}\sqrt{2}\sqrt{5}} \\ &= -\frac{2}{\sqrt{15}}.\end{aligned}$$

Solution 1.2

We must show that the basis is orthonormal, that is

$$e_i\cdot e_j = \delta_{ij}.$$

By direct calculation

$$e_1\cdot e_1 = \frac{1}{6}(1+4+1) = 1,$$

$$e_1\cdot e_2 = \frac{1}{\sqrt{48}}(-2+0+2) = 0,$$

$$e_1\cdot e_3 = \frac{1}{\sqrt{18}}(1-2+1) = 0,$$

$$e_2\cdot e_2 = \frac{1}{8}(4+0+4) = 1,$$

$$e_2\cdot e_3 = \frac{1}{\sqrt{24}}(-2+0+2) = 0,$$

$$e_3\cdot e_3 = \frac{1}{3}(1+1+1) = 1.$$

Orthonormal expansion requires the coefficients:

$$\mathbf{v}\cdot e_1 = \frac{1}{\sqrt{6}}(6+2-1) = \frac{7}{\sqrt{6}},$$

$$\mathbf{v}\cdot e_2 = \frac{1}{\sqrt{8}}(-12+0-2) = \frac{-14}{\sqrt{8}} = \frac{-7}{\sqrt{2}},$$

$$\mathbf{v}\cdot e_3 = \frac{1}{\sqrt{3}}(6-1-1) = \frac{4}{\sqrt{3}}.$$

Thus, with a check:

$$\begin{aligned}\mathbf{v} &= \frac{7}{\sqrt{6}}e_1 - \frac{14}{\sqrt{8}}e_2 + \frac{4}{\sqrt{3}}e_3 \\ &= \tfrac{1}{6}(7,14,7) - \tfrac{1}{8}(-28,0,28) + \tfrac{1}{3}(4,-4,4) \\ &= \tfrac{1}{24}(28+84+32, 56+0-32, 28-84+32) \\ &= \tfrac{1}{24}(144,24,-24) \\ &= (6,1,-1).\end{aligned}$$

Solution 1.3

(a) From the definition of cross product:

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} U_1(\mathbf{p}) & U_2(\mathbf{p}) & U_3(\mathbf{p}) \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= U_1(\mathbf{p})\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - U_2(\mathbf{p})\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + U_3(\mathbf{p})\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}.$$

Hence

$$\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}$$

$$= u_1\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

(b) We apply the properties of the dot product and the rule about interchanging rows changing the sign.

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} &= \mathbf{v}\times\mathbf{w}\cdot\mathbf{u} \\ &= -\mathbf{v}\times\mathbf{u}\cdot\mathbf{w} \\ &= -(-\mathbf{u}\times\mathbf{v}\cdot\mathbf{w}) \\ &= \mathbf{u}\times\mathbf{v}\cdot\mathbf{w}.\end{aligned}$$

Solution 1.4

For the first part, if $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent, then the angle between them is 0 or π. Hence the sine of the angle is 0 and the length of the cross product is 0. Provided that neither vector is zero, each step in the argument is reversible and the result follows. If either is zero, then the vectors are automatically linearly dependent.

The area property follows from

$$\|\mathbf{v}\times\mathbf{w}\| = \|\mathbf{v}\|\,\|\mathbf{w}\|\sin\theta,$$

which is the area of the specified parallelogram.

Solution 1.5

Since we are dealing with a frame, $e_2\times e_3$ is perpendicular to e_2 and to e_3 and so is a multiple of e_1. Also

$$\begin{aligned}\|e_2\times e_3\| &= \|e_2\|\,\|e_3\|\sin(\pi/2) \\ &= 1.\end{aligned}$$

Thus,

$$e_2\times e_3 = \pm e_1$$

and

$$e_1\cdot e_2\times e_3 = \pm e_1\cdot e_1 = \pm 1.$$

Since the definition of orthogonal matrix is equivalent to saying that the rows (or columns) form a frame, the above shows that the determinant of such a matrix is ± 1.

Solution 1.6

Let

$$\mathbf{v}_1 = (\mathbf{v}\cdot\mathbf{u})\,\mathbf{u} \text{ and } \mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1.$$

Then

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 \text{ and } \mathbf{v}_1\cdot\mathbf{v} = \mathbf{v}_1\cdot\mathbf{v}_1 + \mathbf{v}_1\cdot\mathbf{v}_2.$$

Now, substituting for $\mathbf{v}_1$ in the first two occurrences above,

$$\begin{aligned}
&(\mathbf{v}\cdot\mathbf{u})\,\mathbf{u}\cdot\mathbf{v} = ((\mathbf{v}\cdot\mathbf{u})\,\mathbf{u})\cdot((\mathbf{v}\cdot\mathbf{u})\,\mathbf{u}) + \mathbf{v}_1\cdot\mathbf{v}_2\\
&\Rightarrow (\mathbf{v}\cdot\mathbf{u})^2 = (\mathbf{v}\cdot\mathbf{u})^2\mathbf{u}\cdot\mathbf{u} + \mathbf{v}_1\cdot\mathbf{v}_2\\
&\Rightarrow (\mathbf{v}\cdot\mathbf{u})^2 = (\mathbf{v}\cdot\mathbf{u})^2 + \mathbf{v}_1\cdot\mathbf{v}_2 \quad (\mathbf{u} \text{ is a unit vector})\\
&\Rightarrow 0 = \mathbf{v}_1\cdot\mathbf{v}_2.
\end{aligned}$$

Thus $\mathbf{v}$ has an expression of the given form.

Conversely, $\mathbf{v}_1$ is uniquely defined by

$$\mathbf{v}_1 = (\mathbf{v}\cdot\mathbf{u})\,\mathbf{u},$$

hence so is

$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1.$$

Solution 2.1

(a) Finding the derivatives of the coordinate functions of α involves repeated use of the Leibniz property. First, the velocity.

$$\alpha'(t) = (\cos t - t\sin t, \sin t + t\cos t, 1)_{\alpha(t)}.$$

Note that the velocity has been given as a tangent vector based at $\alpha(t)$. Next, the speed, this is the norm of $\alpha'(t)$.

$$\begin{aligned}
\|\alpha'(t)\|^2 &= (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1\\
&= \cos^2 t + t^2\sin^2 t - 2t\cos t\sin t\\
&\quad + \sin^2 t + t^2\cos^2 t + 2t\sin t\cos t + 1\\
&= 2 + t^2;
\end{aligned}$$

so

$$\|\alpha'(t)\| = \sqrt{2+t^2}.$$

Finally, the acceleration.

$$\alpha''(t) = (-2\sin t - t\cos t, 2\cos t - t\sin t, 0)_{\alpha(t)}.$$

(b) Since $\alpha(0) = (0,0,0)$, the route of the curve passes through the origin. At $t = 0$, we have the velocity given by

$$\alpha'(0) = (1,0,1)_{(0,0,0)}$$

and so the speed is $\sqrt{2}$. Finally, the acceleration is

$$\alpha''(0) = (0,2,0)_{(0,0,0)}.$$

Solution 2.2

(a) We apply the technique outlined in the comments on the proof of Theorem 2.1.

We first find the velocity and hence the speed. Here, as later, we shall omit the point of application of the velocity.

$$\begin{aligned}
\alpha'(t) &= (\sinh t, \cosh t, 1).\\
\|\alpha'(t)\| &= \sqrt{\sinh^2 t + \cosh^2 t + 1}\\
&= \sqrt{2\cosh^2 t} \quad (\text{using } 1 + \sinh^2 t = \cosh^2 t)\\
&= \sqrt{2}\cosh t.
\end{aligned}$$

Now

$$\begin{aligned}
s(t) &= \int_0^t \sqrt{2}\cosh u\,du\\
&= \sqrt{2}\,[\sinh u]_0^t\\
&= \sqrt{2}\sinh t.
\end{aligned}$$

Next, we have to find t in terms of s. The method discussed in Part 0 yields

$$t = \log_e\left(\frac{s+\sqrt{s^2+2}}{\sqrt{2}}\right).$$

However, in finding $\alpha(t(s))$, we also need $\sinh t$ and $\cosh t$ and we can obtain these directly from

$$s = \sqrt{2}\sinh t.$$

We have

$$\sinh t = \frac{s}{\sqrt{2}}$$

and

$$\begin{aligned}
\cosh^2 t &= 1 + \sinh^2 t\\
&= 1 + \frac{s^2}{2}\\
&= \frac{2+s^2}{2}.
\end{aligned}$$

Hence

$$\cosh t = \frac{\sqrt{2+s^2}}{\sqrt{2}}.$$

Finally

$$\begin{aligned}
\beta(s) &= \alpha(t(s))\\
&= \left(\frac{\sqrt{s^2+2}}{\sqrt{2}}, \frac{s}{\sqrt{2}}, \log_e\left(\frac{s+\sqrt{s^2+2}}{\sqrt{2}}\right)\right).
\end{aligned}$$

Solution 2.3

To show that the route of the curve passes through the specified points, we look at the first coordinate. If the route is to pass through $(2,1,0)$, it must be for $t = 1$. Checking:

$$\alpha(1) = (2,1,0).$$

Similarly,

$$\alpha(2) = (4,4,\log_e 2).$$

The velocity is given by

$$\alpha'(t) = \left(2, 2t, \frac{1}{t}\right), \quad t > 0,$$

and the speed by

$$\begin{aligned}
\|\alpha'(t)\| &= \sqrt{4 + 4t^2 + \frac{1}{t^2}}\\
&= \sqrt{\left(2t + \frac{1}{t}\right)^2}\\
&= 2t + \frac{1}{t}.
\end{aligned}$$

Hence, the arc-length from $t = 1$ to $t = 2$ is

$$\begin{aligned}
\int_1^2 \left(2u + \frac{1}{u}\right)du &= \left[u^2 + \log_e u\right]_1^2\\
&= 4 + \log_e 2 - 1 - 0\\
&= 3 + \log_e 2.
\end{aligned}$$

Solution 2.4

The easiest approach to this problem is probably to use coordinate functions. Suppose that

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$$

with similar notation for β.

The given information amounts to saying that

$$\alpha_i'(t) = \beta_i'(t), \quad i = 1, 2, 3.$$

Now, all the coordinate functions are simply ordinary, real functions. Two such functions have equal derived functions if, and only if, they differ by a constant. Thus, there must be constants c_1, c_2 and c_3 such that

$$\beta_i(t) = \alpha_i(t) + c_i, \quad i = 1, 2, 3.$$

If we define $\mathbf{p}$ to be the point

$$\mathbf{p} = (c_1, c_2, c_3),$$

then we have shown that

$$\beta(t) = \alpha(t) + \mathbf{p}.$$

This means that the route of β is the route of α translated by the vector $\mathbf{p}$.

Solution 3.1

The easiest approach is to follow the layout of the definitions of the Frenet apparatus. First, though, we check that β is indeed unit speed.

$$\begin{aligned}
\beta'(s) &= (-\tfrac{4}{5}\sin s, -\cos s, \tfrac{3}{5}\sin s)\\
\|\beta'(s)\|^2 &= \tfrac{16}{25}\sin^2 s + \cos^2 s + \tfrac{9}{25}\sin^2 s\\
&= \sin^2 s + \cos^2 s\\
&= 1.
\end{aligned}$$

Thus β is unit speed.

Now we apply the definitions of the Frenet apparatus.

$$\begin{aligned}
T &= \beta'(s)\\
&= (-\tfrac{4}{5}\sin s, -\cos s, \tfrac{3}{5}\sin s);\\
T' &= (-\tfrac{4}{5}\cos s, \sin s, \tfrac{3}{5}\cos s),\\
\kappa &= \|T'\|\\
&= \sqrt{\tfrac{16}{25}\cos^2 s + \sin^2 s + \tfrac{9}{25}\cos^2 s}\\
&= \sqrt{\cos^2 s + \sin^2 s} = 1;\\
N &= \frac{T'}{\kappa}\\
&= (-\tfrac{4}{5}\cos s, \sin s, \tfrac{3}{5}\cos s);\\
B &= T \times N\\
&= (-\tfrac{3}{5}, 0, -\tfrac{4}{5});\\
B' &= 0;\\
\tau &= -B' \cdot N = 0.
\end{aligned}$$

The results above show that β is a plane curve ($\tau = 0$), and has constant curvature 1. Thus (the route of) β is a circle of radius 1.

There are a number of methods of finding the centre. Perhaps the most elementary is to suppose that the centre is $\mathbf{p} = (p_1, p_2, p_3)$ and write down the equations that result from expressing the fact that three selected points on the curve are all 1 unit from $\mathbf{p}$. If we take the points

$$\beta(0) = (\tfrac{4}{5}, 1, -\tfrac{3}{5}),$$

$$\beta(\pi/2) = (0, 0, 0)$$

and

$$\beta(\pi) = (-\tfrac{4}{5}, 1, \tfrac{3}{5})$$

then this approach yields the three equations

$$\begin{aligned}
(p_1 - 4/5)^2 + (p_2 - 1)^2 + (p_3 + 3/5)^2 &= 1,\\
p_1^2 + p_2^2 + p_3^2 &= 1,\\
(p_1 + 4/5)^2 + (p_2 - 1)^2 + (p_3 - 3/5)^2 &= 1.
\end{aligned}$$

Expanding the first and third equations and subtracting the second from both results yields

$$\begin{aligned}
-(8/5)p_1 - 2p_2 + (6/5)p_3 + 2 &= 0,\\
(8/5)p_1 - 2p_2 - (6/5)p_3 + 2 &= 0.
\end{aligned}$$

Adding, we have

$$-4p_2 + 4 = 0,$$

and so $p_2 = 1$.

Substituting in the second of the original equations gives

$$p_1^2 + p_3^2 = 0.$$

Since squares are non-negative, this forces $p_1 = p_3 = 0$ and so the centre is $\mathbf{p} = (0, 1, 0)$.

Alternatively, we can use the remark on page 62 of *O'Neill*, just after the proof of Corollary 3.5. There it says that the unit normal N points towards the centre. Thus, we can start at any point and move one unit along the normal. If we start at

$$\beta(0) = (\tfrac{4}{5}, 0, -\tfrac{3}{5})$$

and travel one unit along

$$N(0) = (-\tfrac{4}{5}, 0, \tfrac{3}{5}),$$

we arrive at

$$\beta(0) + N(0) = (0, 1, 0)$$

as before.

We can confirm that this centre is correct by considering

$$\beta(s) - (0, 1, 0) = (\tfrac{4}{5}\cos s, -\sin s, -\tfrac{3}{5}\cos s).$$

We have

$$\begin{aligned}
\|\beta(s) - (0, 1, 0)\|^2 &= \frac{16}{25}\cos^2 s + \sin^2 s + \frac{9}{25}\cos^2 s\\
&= \cos^2 s + \sin^2 s = 1.
\end{aligned}$$

Thus $\beta(s)$ is a constant distance 1 from the point $(0, 1, 0)$. Thus the route is a circle, radius 1, centre $(0, 1, 0)$.

Solution 3.2

We proceed as in the first part of the last solution.

$$\begin{aligned}
\beta'(s) &= \left(\frac{3}{2}\frac{(1+s)^{1/2}}{3}, -\frac{3}{2}\frac{(1-s)^{1/2}}{3}, \frac{1}{\sqrt{2}}\right)\\
&= \left(\frac{(1+s)^{1/2}}{2}, -\frac{(1-s)^{1/2}}{2}, \frac{1}{\sqrt{2}}\right);
\end{aligned}$$

$$\|\beta'(s)\|^2 = \tfrac{1}{4}(1+s) + \tfrac{1}{4}(1-s) + \tfrac{1}{2} = 1.$$

Hence, β is unit speed. It follows that

$$\begin{aligned}
T &= \beta'(s)\\
&= \left(\frac{(1+s)^{1/2}}{2}, -\frac{(1-s)^{1/2}}{2}, \frac{1}{\sqrt{2}}\right);\\
T' &= \left(\tfrac{1}{4}(1+s)^{-1/2}, \tfrac{1}{4}(1-s)^{-1/2}, 0\right),
\end{aligned}$$

$$
\begin{aligned}
\kappa &= \|T'\| \\
&= \tfrac{1}{4}\sqrt{(1+s)^{-1}+(1-s)^{-1}} \\
&= \tfrac{1}{4}\sqrt{\frac{1-s+1+s}{(1+s)(1-s)}} \\
&= \tfrac{1}{4}\sqrt{\frac{2}{1-s^2}} \\
&= \frac{1}{2\sqrt{2}\sqrt{1-s^2}};
\end{aligned}
$$

$$
\begin{aligned}
N &= \frac{T'}{\kappa} \\
&= 4\sqrt{\frac{1-s^2}{2}}\left(\tfrac{1}{4}(1+s)^{-1/2}, \tfrac{1}{4}(1-s)^{-1/2}, 0\right); \\
&= \frac{1}{\sqrt{2}}\left((1-s)^{1/2}, (1+s)^{1/2}, 0\right);
\end{aligned}
$$

$$
\begin{aligned}
B &= T \times N \\
&= \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}(1+s)^{1/2}, \frac{1}{\sqrt{2}}(1-s)^{1/2}, \frac{1+s}{2}+\frac{1-s}{2}\right) \\
&= \tfrac{1}{2}\left(-(1+s)^{1/2}, (1-s)^{1/2}, \sqrt{2}\right);
\end{aligned}
$$

$$
\begin{aligned}
B' &= \tfrac{1}{2}\left(-\tfrac{1}{2}(1+s)^{-1/2}, -\tfrac{1}{2}(1-s)^{-1/2}, 0\right) \\
&= \tfrac{1}{4}\left(-(1+s)^{-1/2}, -(1-s)^{-1/2}, 0\right);
\end{aligned}
$$

$$
\begin{aligned}
\tau &= -B' \cdot N \\
&= -\frac{1}{4\sqrt{2}}\left(-\sqrt{\frac{1-s}{1+s}} - \sqrt{\frac{1+s}{1-s}}\right) \\
&= \frac{1}{4\sqrt{2}}\left(\sqrt{\frac{1-s}{1+s}} + \sqrt{\frac{1+s}{1-s}}\right) \\
&= \frac{1}{4\sqrt{2}}\left(\frac{(1-s)+(1+s)}{\sqrt{(1+s)(1-s)}}\right) \\
&= \frac{1}{4\sqrt{2}}\frac{2}{\sqrt{1-s^2}} \\
&= \frac{1}{2\sqrt{2}\sqrt{1-s^2}}.
\end{aligned}
$$

Note that, since $\tau = \kappa$, the ratio

$$\frac{\tau}{\kappa} = 1$$

is constant. As will be shown in the next section, this is sufficient to show that β is a cylindrical helix.

Solution 3.3

We simply compute the three cross products and show that they give the right-hand sides of the respective Frenet formulas and, hence, the three derivatives.

$$
\begin{aligned}
A \times T &= (\tau T + \kappa B) \times T \\
&= \tau 0 + \kappa N \\
&= T';
\end{aligned}
$$

$$
\begin{aligned}
A \times N &= (\tau T + \kappa B) \times N \\
&= \tau B + \kappa(-T) \\
&= N';
\end{aligned}
$$

$$
\begin{aligned}
A \times B &= (\tau T + \kappa B) \times B \\
&= \tau(-N) + \kappa 0 \\
&= B'.
\end{aligned}
$$

Solution 3.4

We write down the three requirements

$$\gamma(0) = \beta(0), \quad \gamma'(0) = \beta'(0), \quad \gamma''(0) = \beta''(0)$$

and see what can be deduced by use of the Frenet formulas.

First, we calculate the values of γ and its first two derivatives at 0:

$$
\begin{aligned}
\gamma(0) &= \mathbf{c} + r\mathbf{e}_1, \\
\gamma'(s) &= -\sin(s/r)\mathbf{e}_1 + \cos(s/r)\mathbf{e}_2, \\
\gamma'(0) &= \mathbf{e}_2 \\
\gamma''(s) &= -(1/r)\cos(s/r)\mathbf{e}_1 - (1/r)\cos(s/r)\mathbf{e}_2, \\
\gamma''(0) &= -(1/r)\mathbf{e}_1.
\end{aligned}
$$

Next, we do the same for the derivatives of β.

$$
\begin{aligned}
\beta'(s) &= T(s), \\
\beta'(0) &= T(0), \\
\beta''(s) &= T'(s) \\
&= \kappa(s)N(s), \\
\beta''(0) &= \kappa(0)N(0).
\end{aligned}
$$

Now we equate the various values that we have obtained.

$$
\begin{aligned}
\mathbf{c} + r\mathbf{e}_1 &= \beta(0), \\
\mathbf{e}_2 &= T(0), \\
-(1/r)\mathbf{e}_1 &= \kappa(0)N(0).
\end{aligned}
$$

We are aiming to find r, $\mathbf{e}_1$ and $\mathbf{e}_2$ in terms of the Frenet apparatus of β. Taking norms in the third equation above gives

$$|(1/r)| = |\kappa(0)|, \quad (\|N\| = \|\mathbf{e}_1\| = 1).$$

Since r and κ are positive, we have

$$r = 1/\kappa(0).$$

The third equation also shows that $\mathbf{e}_1$ and $N(0)$ are unit vectors in opposite directions, that is,

$$\mathbf{e}_1 = -N(0).$$

The second equation shows directly that

$$\mathbf{e}_2 = T(0).$$

Finally,

$$
\begin{aligned}
\mathbf{c} &= \beta(0) - r\mathbf{e}_1 \\
&= \beta(0) - (1/\kappa(0))(-N(0)) \\
&= \beta(0) + (1/\kappa(0))N(0).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\gamma(s) = \beta(0) &+ (1/\kappa(0))(1 - \cos(s\kappa(0)))N(0) \\
&+ (1/\kappa(0))\sin(s\kappa(0))T(0).
\end{aligned}
$$

This shows that γ lies in the T–N plane at $\beta(0)$, that is, the osculating plane at $\beta(0)$.

Solution 3.5

(a) There are two approaches suggested by what we have done. The first uses the same techniques as were used for the Frenet formulas. We express N' using orthonormal expansion:

$$N' = (N' \cdot T)T + (N' \cdot N)N.$$

Now we use the fact that N and T are orthogonal unit vectors, together with the Leibniz property.

$$
\begin{aligned}
& N \cdot N = 1 \\
\Rightarrow\ & 2N' \cdot N = 0 \\
\Rightarrow\ & N' \cdot N = 0.
\end{aligned}
$$

$$\begin{aligned} &T\cdot N = 0\\ &\Rightarrow T'\cdot N + T\cdot N' = 0\\ &\Rightarrow N'\cdot T = -T'\cdot N.\end{aligned}$$

Since

$$\begin{aligned} -T'\cdot N &= -(\kappa N)\cdot N\\ &= -\kappa,\end{aligned}$$

substituting for $N'\cdot T$ and $N'\cdot N$ in the orthonormal expansion gives

$$N' = -\kappa T$$

as required.

We could also tackle the problem directly. Using the notation in the question, and the fact that $T' = \kappa N$, we have

$$\begin{aligned} T' &= (x'', y'')\\ &= \kappa(-y', x')\\ &= (-\kappa y', \kappa x').\end{aligned}$$

Thus $x'' = -\kappa y'$ and $y'' = \kappa x'$.

Hence

$$\begin{aligned} N' &= (-y'', x'')\\ &= (-\kappa x', -\kappa y')\\ &= -\kappa(x', y')\\ &= -\kappa T.\end{aligned}$$

(b) Note that, as *O'Neill* says in the footnote, ϕ is the angle between T and the x-axis.

Probably the easiest way to tackle this is to express T' and N in terms of ϕ. Because of the definition of N, we have

$$N = -\sin\phi\, U_1 + \cos\phi\, U_2.$$

Differentiating the expression given for T with respect to s gives

$$T' = -\sin\phi\,\phi'(s)\,U_1 + \cos\phi\,\phi'(s)\,U_2.$$

(We have used the chain rule here.)

Comparing T' and κN, gives

$$\kappa = \phi'.$$

Intuitively, this result is reasonable. Curvature is the rate of turning of the tangent, which is precisely what ϕ' measures.

Solution 4.1

We carry out the calculations as indicated by Theorem 4.3. We calculate the derivatives of α, the cross and triple products and then use the formulas from the theorem.

$$\begin{aligned} \alpha'(t) &= (2, 2t, t^2);\\ \alpha''(t) &= (0, 2, 2t);\\ \alpha'''(t) &= (0, 0, 2).\\ \alpha'(t)\times\alpha''(t) &= (2t^2, -4t, 4),\\ \|\alpha'(t)\times\alpha''(t)\| &= \sqrt{4t^4 + 16t^2 + 16}\\ &= \sqrt{4(t^2+2)^2}\\ &= 2(2+t^2).\end{aligned}$$

$$\alpha'(t)\times\alpha''(t)\cdot\alpha'''(t) = 8.$$

Hence

$$\begin{aligned} v &= \|\alpha'(t)\|\\ &= \sqrt{4 + 4t^2 + t^4}\\ &= \sqrt{(2+t^2)^2}\\ &= 2 + t^2.\end{aligned}$$

It follows that

$$\begin{aligned} T &= \frac{1}{v}\alpha'(t)\\ &= \frac{1}{2+t^2}(2, 2t, t^2).\end{aligned}$$

Next

$$\kappa v^3 = \|\alpha'(t)\times\alpha''(t)\|.$$

Hence

$$\begin{aligned} \kappa &= \frac{2(2+t^2)}{v^3}\\ &= \frac{2(2+t^2)}{(2+t^2)^3}\\ &= \frac{2}{(2+t^2)^2}.\end{aligned}$$

Now B:

$$\begin{aligned} B &= \frac{\alpha'(t)\times\alpha''(t)}{\|\alpha'(t)\times\alpha''(t)\|}\\ &= \frac{1}{2(2+t^2)}(2t^2, -4t, 4)\\ &= \frac{1}{2+t^2}(t^2, -2t, 2).\end{aligned}$$

$$\begin{aligned} N &= B\times T\\ &= \frac{1}{(2+t^2)^2}\left(-2t(t^2+2), 4-t^4, 2t(t^2+2)\right)\\ &= \frac{1}{2+t^2}(-2t, 2-t^2, 2t).\end{aligned}$$

Finally,

$$\begin{aligned} \tau &= \frac{\alpha'(t)\times\alpha''(t)\cdot\alpha'''(t)}{\|\alpha'(t)\times\alpha''(t)\|^2}\\ &= \frac{8}{4(2+t^2)^2} = \frac{2}{(2+t^2)^2}.\end{aligned}$$

Note that $\tau/\kappa = 1$, so that the curve is a cylindrical helix.

Solution 4.2

We approach this in the same way as the last question.

$$\begin{aligned} \alpha'(t) &= (\sinh t, \cosh t, 1),\\ \alpha''(t) &= (\cosh t, \sinh t, 0),\\ \alpha'''(t) &= (\sinh t, \cosh t, 0).\end{aligned}$$

$$\begin{aligned} \alpha'(t)\times\alpha''(t) &= (-\sinh t, \cosh t, \sinh^2 t - \cosh^2 t)\\ &= (-\sinh t, \cosh t, -1),\\ &\quad(\text{because } \cosh^2 t - \sinh^2 t = 1);\end{aligned}$$

$$\begin{aligned} \|\alpha'(t)\times\alpha''(t)\| &= \sqrt{\sinh^2 t + \cosh^2 t + 1}\\ &= \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t.\end{aligned}$$

$$\begin{aligned} \alpha'(t)\times\alpha''(t)\cdot\alpha'''(t) &= -\sinh^2 t + \cosh^2 t\\ &= 1.\end{aligned}$$

$$\begin{aligned} v &= \|\alpha'(t)\|\\ &= \sqrt{\sinh^2 t + \cosh^2 t + 1}\\ &= \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t.\end{aligned}$$

$$\begin{aligned}T &= \frac{1}{v}\alpha'(t)\\ &= \frac{1}{\sqrt{2}\cosh t}(\sinh t, \cosh t, 1).\\ \kappa &= \frac{\|\alpha'(t)\times\alpha''(t)\|}{v^3}\\ &= \frac{\sqrt{2}\cosh t}{2\sqrt{2}\cosh^3 t} = \frac{1}{2\cosh^2 t}.\\ B &= \frac{\alpha'(t)\times\alpha''(t)}{\|\alpha'(t)\times\alpha''(t)\|}\\ &= \frac{1}{\sqrt{2}\cosh t}(-\sinh t, \cosh t, -1).\\ N &= B\times T\\ &= \frac{1}{2\cosh^2 t}(2\cosh t, 0, 2\sinh t\cosh t)\\ &= \frac{1}{\cosh t}(1, 0, \sinh t).\\ \tau &= \frac{\alpha'(t)\times\alpha''(t)\cdot\alpha'''(t)}{\|\alpha'(t)\times\alpha''(t)\|^2}\\ &= \frac{1}{2\cosh^2 t}.\end{aligned}$$

Once again, this is a cylindrical helix.

To complete the question, we must find the arc-length function. We have

$$\begin{aligned}s(t) &= \int_0^t v(u)\,du\\ &= \int_0^t \sqrt{2}\cosh u\,du\\ &= \left[\sqrt{2}\sinh u\right]_0^t = \sqrt{2}\sinh t.\end{aligned}$$

Since

$$\cosh^2 t = 1 + \sinh^2 t = 1 + (s/\sqrt{2})^2 = 1 + s^2/2,$$

we have

$$\begin{aligned}\kappa &= \tau\\ &= \frac{1}{2(1+s^2/2)}\\ &= \frac{1}{2+s^2}.\end{aligned}$$

Solution 4.3

Following the suggestion in the question, all derivatives are evaluated at $t = 0$ before calculating cross products etc.

$$\begin{aligned}\alpha'(t) &= (\cos t - t\sin t, \sin t + t\cos t, 1),\\ \alpha''(t) &= (-2\sin t - t\cos t, 2\cos t - t\sin t, 0),\\ \alpha'''(t) &= (-3\cos t + t\sin t, -3\sin t - t\cos t, 0).\end{aligned}$$

$$\begin{aligned}\alpha'(0) &= (1, 0, 1),\\ \alpha''(0) &= (0, 2, 0),\\ \alpha'''(t) &= (-3, 0, 0).\\ \alpha'(0)\times\alpha''(0) &= (-2, 0, 2)\\ \|\alpha'(0)\times\alpha''(0)\| &= \sqrt{4+0+4}\\ &= 2\sqrt{2}.\end{aligned}$$

$$\alpha'(0)\times\alpha''(0)\cdot\alpha'''(0) = 6.$$

$$\begin{aligned}v(0) &= \sqrt{1+0+1}\\ &= \sqrt{2}.\end{aligned}$$

$$\begin{aligned}T(0) &= \frac{\alpha'(0)}{v(0)}\\ &= \frac{1}{\sqrt{2}}(1, 0, 1).\\ \kappa(0) &= \frac{\|\alpha'(0)\times\alpha''(0)\|}{v(0)^3} = \frac{2\sqrt{2}}{2\sqrt{2}} = 1.\\ B(0) &= \frac{\alpha'(0)\times\alpha''(0)}{\|\alpha'(0)\times\alpha''(0)\|}\\ &= \frac{1}{2\sqrt{2}}(-2, 0, 2)\\ &= \frac{1}{\sqrt{2}}(-1, 0, 1).\\ N &= B\times T\\ &= \tfrac{1}{2}(0, 2, 0) = (0, 1, 0).\\ \tau(0) &= \frac{\alpha'(0)\times\alpha''(0)\cdot\alpha'''(0)}{\|\alpha'(0)\times\alpha''(0)\|^2}\\ &= \tfrac{6}{8} = \tfrac{3}{4}.\end{aligned}$$

Solution 4.4

The solution below follows the hint given.

$$\begin{aligned}\alpha'(t) &= cT,\\ \alpha''(t) &= cT' \quad (c \text{ constant})\\ &= c(c\kappa N) \quad \text{(Frenet)}\\ &= c^2\kappa N.\\ \alpha'''(t) &= c^2\kappa' N + c^2\kappa N'\\ &\qquad \text{(we may not assume that } \kappa \text{ is constant)}\\ &= c^2\kappa' N + c^2\kappa(-c\kappa T + c\tau B) \quad \text{(Frenet)}\\ &= -c^3\kappa T + c^2\kappa' N + c^3\kappa\tau B.\end{aligned}$$

The first line immediately gives the first result:

$$T = \frac{\alpha'}{c}.$$

The expression for α'' shows that it and N are in the same direction and so

$$N = \frac{\alpha''}{\|\alpha''\|}.$$

It follows that

$$\begin{aligned}B &= T\times N\\ &= \frac{\alpha'}{c}\times\frac{\alpha''}{\|\alpha''\|}\\ &= \frac{\alpha'\times\alpha''}{c\|\alpha''\|}.\end{aligned}$$

By taking the norm of α'' we obtain

$$\|\alpha''\| = c^2\kappa$$

and, hence,

$$\kappa = \frac{\|\alpha''\|}{c^2}.$$

Now we calculate the triple product.

$$\begin{aligned}\alpha'\times\alpha'' &= cT\times c^2\kappa N\\ &= c^3\kappa T\times N\\ &= c^3\kappa B.\\ \alpha'\times\alpha''\cdot\alpha''' &= 0 + 0 + c^6\kappa^2\tau.\end{aligned}$$

Since

$$\begin{aligned}c^6\kappa^2 &= c^6\frac{\|\alpha''\|^2}{c^4}\\ &= c^2\|\alpha''\|^2,\end{aligned}$$

we have

$$\begin{aligned}\tau &= \frac{\alpha' \times \alpha'' \cdot \alpha'''}{c^6 \kappa^2}\\ &= \frac{\alpha' \times \alpha'' \cdot \alpha'''}{c^2 \|\alpha''\|^2}.\end{aligned}$$

Solution 4.5

First of all we show that the route of γ passes through $\alpha(0)$ and then that the tangent to γ is always perpendicular to $\mathbf{u}$.

$$\begin{aligned}\gamma(0) &= \alpha(0) - s(0)\cos\vartheta\,\mathbf{u}\\ &= \alpha(0)\end{aligned}$$

since $s(0) = 0$ from the information given.

Now

$$\begin{aligned}\gamma(t) &= \alpha(t) - s(t)\cos\vartheta\,\mathbf{u},\\ \gamma'(t) &= \alpha'(t) - s'(t)\cos\vartheta\,\mathbf{u}\\ &= vT - v\cos\vartheta\,\mathbf{u}.\\ \gamma'(t)\cdot\mathbf{u} &= vT\cdot\mathbf{u} - v\cos\vartheta\,\mathbf{u}\cdot\mathbf{u}\\ &= v\cos\vartheta - v\cos\vartheta\\ &= 0.\end{aligned}$$

In the above we have used the fact that $\mathbf{u}$ is a constant unit vector, that ϑ is constant and that $T\cdot\mathbf{u} = \cos\vartheta$.

We have now shown that γ passes through $\alpha(0)$ and its tangent is always orthogonal to $\mathbf{u}$. That is sufficient to prove the assertion in part (a).

We now calculate the curvature of γ; as suggested we deal only with the case that α is unit speed. We also make use of the expression for $\mathbf{u}$:

$$\mathbf{u} = \cos\vartheta\,T + \sin\vartheta\,B.$$

From the above we have

$$\begin{aligned}\gamma'(t) &= vT - v\cos\vartheta\,\mathbf{u}\\ &= T - \cos\vartheta\,\mathbf{u}, \quad \text{(unit speed)}.\\ \|\gamma'(t)\|^2 &= (T - \cos\vartheta\,\mathbf{u})\cdot(T - \cos\vartheta\,\mathbf{u})\\ &= T\cdot T - 2\cos\vartheta\,\mathbf{u}\cdot T + \cos^2\vartheta\,\mathbf{u}\cdot\mathbf{u}\\ &= 1 - \cos^2\vartheta\\ &= \sin^2\vartheta.\\ \|\gamma'(t)\| &= \sin\vartheta.\\ \gamma''(t) &= T' - 0\\ &= \kappa N.\end{aligned}$$

$$\begin{aligned}\gamma'(t)\times\gamma''(t) &= (T - \cos\vartheta\,\mathbf{u})\times\kappa N\\ &= \kappa(T\times N - \cos\vartheta\,\mathbf{u}\times N)\\ &= \kappa\,(B - \cos\vartheta(\cos\vartheta\,T + \sin\vartheta\,B)\times N)\\ &= \kappa\left(B(1-\cos^2\vartheta) + \cos\vartheta\sin\vartheta\,T\right)\\ &= \kappa\sin\vartheta(\cos\vartheta\,T + \sin\vartheta\,B)\\ &= \kappa\sin\vartheta\,\mathbf{u}.\\ \|\gamma'(t)\times\gamma''(t)\| &= \kappa\sin\vartheta.\end{aligned}$$

$$\kappa_\gamma = \frac{\kappa\sin\vartheta}{\sin^3\vartheta} = \frac{\kappa}{\sin^2\vartheta}.$$

Solution 4.6

We apply the results of the previous exercise.

First, because τ/κ is clearly a constant, the curve is a cylindrical helix.

The cross-section curve has curvature

$$\frac{\kappa}{\sin^2\vartheta}$$

which is, from the given information, a constant. The cross-section curve is, therefore, a plane curve of constant curvature. That is, (part of) a circle. The cylinder on which the curve lies has a circular cross-section and the curve is, therefore, a circular helix.

Solution 4.7

Making use of the expression derived in *O'Neill* for σ'', we have

$$\begin{aligned}\sigma''' &= -2\kappa\kappa' T - \kappa^2 T' + \kappa'' N + \kappa' N' + (\kappa\tau)' B + \kappa\tau B'\\ &= -2\kappa\kappa' T - \kappa^3 N + \kappa'' N + \kappa'(-\kappa T + \tau B)\\ &\quad + (\kappa'\tau + \kappa\tau')B - \kappa\tau^2 N\\ &= -3\kappa\kappa' T + (\kappa'' - \kappa^3 - \kappa\tau^2)N + (2\kappa'\tau + \kappa\tau')B.\end{aligned}$$

$$\begin{aligned}\sigma'\times\sigma''\cdot\sigma''' &= (\kappa^2\tau)(-3\kappa\kappa') + \kappa^3(2\kappa'\tau + \kappa\tau')\\ &= \kappa^3(\kappa\tau' - \kappa'\tau).\end{aligned}$$

Since

$$\left(\frac{\tau}{\kappa}\right)' = \frac{\kappa\tau' - \kappa'\tau}{\kappa^2},$$

we have

$$\begin{aligned}\sigma'\times\sigma''\cdot\sigma''' &= \kappa^3\kappa^2\left(\frac{\tau}{\kappa}\right)'\\ &= \kappa^5\left(\frac{\tau}{\kappa}\right)'.\end{aligned}$$

We also have

$$\begin{aligned}\|\sigma'\times\sigma''\| &= \kappa^2\sqrt{\kappa^2+\tau^2}\\ &= \kappa^3\sqrt{1+(\tau/\kappa)^2}.\end{aligned}$$

Thus

$$\begin{aligned}\tau_\sigma &= \frac{\sigma'\times\sigma''\cdot\sigma'''}{\|\sigma'\times\sigma''\|^2}\\ &= \frac{\kappa^5(\tau/\kappa)'}{\kappa^6\,(1+(\tau/\kappa)^2)}\\ &= \frac{(\tau/\kappa)'}{\kappa\,(1+(\tau/\kappa)^2)}.\end{aligned}$$

Solution 4.8

We can apply all the results obtained so far about spherical images.

A unit-speed curve β is a cylindrical helix if, and only if, the ratio τ_β/κ_β is constant. Suppose that this constant is c and the spherical image is σ. Then, if β is a cylindrical helix,

$$\kappa_\sigma = \sqrt{1+c^2}$$

which is constant. Further

$$\tau_\sigma = \frac{(d/ds)(c)}{\kappa_\beta(1+c^2)} = 0.$$

Thus σ is a plane ($\tau = 0$) curve of constant curvature, that is part of a circle.

Conversely, if the spherical image is part of a circle, then

$$\tau_\sigma = 0$$

and the expression for τ_σ shows that

$$\frac{d}{ds}\left(\frac{\tau_\beta}{\kappa_\beta}\right) = 0.$$

It follows that τ_β/κ_β is constant and β is a cylindrical helix.

Solution 5.1

Since the question said 'from first principles', we apply Definition 5.1 directly.

(a) We have

$$\begin{aligned}\nabla_{\mathbf{v}_\mathbf{p}} W &= (W(1+t, 3-t, -1+2t))'(0)\\ &= \left((1+t)^2 U_1(\mathbf{p}+t\mathbf{v}) + (3-t)U_2(\mathbf{p}+t\mathbf{v})\right)'(0)\\ &= (2(1+t)U_1(\mathbf{p}+t\mathbf{v}) + (-1)U_2(\mathbf{p}+t\mathbf{v}))(0)\\ &= 2U_1(\mathbf{p}) - U_2(\mathbf{p}) \quad (= (2,-1,0)_\mathbf{p}).\end{aligned}$$

(b) In this case

$$\begin{aligned}\nabla_{\mathbf{v}_\mathbf{p}} W &= \big((1+t)U_1(\mathbf{p}+t\mathbf{v}) + (1+t)^2 U_2(\mathbf{p}+t\mathbf{v})\\ &\qquad\qquad -(-1+2t)^2 U_3(\mathbf{p}+t\mathbf{v})\big)'(0)\\ &= (U_1(\mathbf{p}+t\mathbf{v}) + 2(1+t)U_2(\mathbf{p}+t\mathbf{v})\\ &\qquad\qquad -4(-1+2t)U_3(\mathbf{p}+t\mathbf{v}))(0)\\ &= U_1(\mathbf{p}) + 2U_2(\mathbf{p}) + 4U_3(\mathbf{p}) \quad (= (1,2,4)_\mathbf{p}).\end{aligned}$$

Solution 5.2

The approach here is to use the linearity properties of the covariant derivative, together with the basic result

$$\nabla_{U_i} f U_j = \frac{\partial f}{\partial x_i} U_j.$$

(a) This is a straightforward application of both varieties of linearity from Corollary 5.4.

$$\begin{aligned}\nabla_V W &= \nabla_{-yU_1 + xU_3} W\\ &= -y\nabla_{U_1} W + x\nabla_{U_3} W \quad \text{(linearity)}\\ &= -y\left(\nabla_{U_1} \cos x\, U_1 + \nabla_{U_1} \sin x\, U_2\right)\\ &\quad + x\left(\nabla_{U_3} \cos x\, U_1 + \nabla_{U_3} \sin x\, U_2\right) \quad \text{(linearity)}\\ &= -y\left(-\sin x\, U_1 + \cos x\, U_2\right) + x(0+0)\\ &= y(\sin x\, U_1 - \cos x\, U_2).\end{aligned}$$

(b) Here we first apply Corollary 5.4(3) and then use the above result.

$$\begin{aligned}\nabla_V z^2 W &= V[z^2]W + z^2 \nabla_V W\\ &= (-yU_1 + xU_3)[z^2]\, W + z^2 y(\sin xU_1 - \cos xU_2)\\ &= (-y \times 0 + x \times 2z)(\cos xU_1 + \sin xU_2)\\ &\quad + z^2 y(\sin xU_1 - \cos xU_2)\\ &= (2xz\cos x + yz^2 \sin x)U_1\\ &\quad + (2xz\sin x - yz^2\cos x)U_2.\end{aligned}$$

(c) We make use of the result from the first part of this question.

$$\begin{aligned}\nabla_V \nabla_V W &= \nabla_V y(\sin x\, U_1 - \cos x\, U_2)\\ &= -y\nabla_{U_1} y(\sin x\, U_1 - \cos x\, U_2)\\ &\quad + x\nabla_{U_3} y(\sin x\, U_1 - \cos x\, U_2)\\ &= -y\left(y(\cos x\, U_1 + \sin x\, U_2) + x(0+0)\right)\\ &= -y^2(\cos x\, U_1 + \sin x\, U_2) \quad (= -y^2 W).\end{aligned}$$

Solution 5.3

Suppose that V has coordinate functions v_i for $i = 1, 2, 3$. That is,

$$V = \sum_{i=1}^{3} v_i U_i.$$

Then, by linearity,

$$\nabla_V X = \sum_{i=1}^{3} v_i \nabla_{U_i} X.$$

But

$$\begin{aligned}\nabla_{U_i} X &= \nabla_{U_i}(x_1U_1 + x_2U_2 + x_3U_3)\\ &= U_i\end{aligned}$$

because

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij}.$$

Hence

$$\nabla_V X = \sum_{i=1}^{3} v_i U_i = V.$$

Solution 5.4

We begin by using the definition to calculate ∇W.

$$\begin{aligned}\nabla W &= d(xy^3)U_1 - d(x^2z^2)U_3\\ &= (y^3\, dx + 3xy^2\, dy)U_1 - (2xz^2\, dx + 2x^2z\, dz)U_3.\end{aligned}$$

Now, using the definition given and the methods of Chapter I, we can do the evaluations.

(a) We have

$$\begin{aligned}\nabla_{\mathbf{v}_\mathbf{p}} W &= (\nabla W)(\mathbf{v}_\mathbf{p})\\ &= \left((y^3\, dx + 3xy^2\, dy)U_1 - (2xz^2\, dx + 2x^2z\, dz)U_3\right)(\mathbf{v}_\mathbf{p})\\ &= (8(1) + 3(-1)(4)(0))U_1(\mathbf{p})\\ &\quad - (2(-1)(1)(1) + 2(1)(-1)(-3))U_3(\mathbf{p})\\ &= 8U_1(\mathbf{p}) - 4U_3(\mathbf{p}).\end{aligned}$$

(b) This time, the same method leads to

$$\nabla_{\mathbf{v}_\mathbf{p}} W = 27U_1(\mathbf{p}) + 12U_3(\mathbf{p}).$$

Solution 5.5

We have

$$\begin{aligned}\alpha(0) &= (1+\sin(0), 3-\sin(0), -1)\\ &= (1, 3, -1).\\ \alpha'(t) &= (\cos t, -\cos t, 2t+2).\\ \alpha'(0) &= (1, -1, 2).\end{aligned}$$

Next

$$W(\alpha(t)) = (1+\sin t)^2 U_1(\alpha(t)) + (3-\sin t)U_2(\alpha(t)).$$

So

$$\begin{aligned}\nabla_{(1,-1,2)_{(1,3,-1)}} W &= (W(\alpha(t)))'(0)\\ &= (2\cos t(1+\sin t)U_1(\alpha(t))\\ &\qquad - \cos t U_2(\alpha(t)))(0)\\ &= 2U_1(1,3,-1) - U_2(1,3,-1).\end{aligned}$$

This is exactly the same result as we obtained in the first solution of this set. Since, using the notation from that solution,

$$\begin{aligned}\alpha(0) &= \mathbf{p}\\ \alpha'(0) &= \mathbf{v}\end{aligned}$$

this is to be expected.

Solution 6.1

Since we are given the alleged frame, all we have to do is check that

$$E_i \cdot E_j = \delta_{ij}.$$

$$E_1 \cdot E_1 = \frac{V \cdot V}{\|V\|^2} = \frac{\|V\|^2}{\|V\|^2} = 1.$$

A similar calculation shows that $E_2 \cdot E_2 = 1$.

Now we show that the vector fields E_1 and E_2 are orthogonal.

$$
\begin{aligned}
E_1 \cdot E_2 &= \frac{E_1 \cdot (W - (W \cdot E_1)E_1)}{\|\tilde{W}\|} \\
&= \frac{(W \cdot E_1) - (W \cdot E_1)(E_1 \cdot E_1)}{\|\tilde{W}\|} \\
&= \frac{(W \cdot E_1) - (W \cdot E_1) \times 1}{\|\tilde{W}\|} \\
&= 0.
\end{aligned}
$$

Since E_1 and E_2 are orthogonal unit vector fields, their cross product E_3 is orthogonal to both of them and

$$\|E_3\| = 1 \times 1 \times \sin(\pi/2) = 1.$$

This completes the proof that the given fields form a frame field.

Solution 6.2

We apply Lemma 6.3, that is orthonormal expansion. We repeatedly use the fact that

$$U_i \cdot U_j = \delta_{ij}.$$

(a) Here

$$
\begin{aligned}
U_1 \cdot E_1 &= U_1 \cdot (\cos\vartheta\, U_1 + \sin\vartheta\, U_2) \\
&= \cos\vartheta, \\
U_1 \cdot E_2 &= U_1 \cdot (-\sin\vartheta\, U_1 + \cos\vartheta\, U_2) \\
&= -\sin\vartheta, \\
U_1 \cdot E_3 &= U_1 \cdot U_3 \\
&= 0. \\
U_1 &= \cos\vartheta\, E_1 - \sin\vartheta\, E_2.
\end{aligned}
$$

(b) We can save quite a bit of effort by noting that the first two terms are just E_1, so the given vector field is

$$E_1 + E_3.$$

Direct calculation of dot products gives exactly the same result, using $\sin^2\vartheta + \cos^2\vartheta = 1$.

(c) Orthonormal expansion gives

$$
\begin{aligned}
(x\, U_1 + y\, U_2 + z\, U_3) \cdot E_1 &= x\cos\vartheta + y\sin\vartheta \\
&= r\cos^2\vartheta + r\sin^2\vartheta \\
&= r, \\
(x\, U_1 + y\, U_2 + z\, U_3) \cdot E_2 &= -x\sin\vartheta + y\cos\vartheta \\
&= r\cos\vartheta\sin\vartheta + r\sin\vartheta\cos\vartheta \\
&= 0, \\
(x\, U_1 + y\, U_2 + z\, U_3) \cdot E_3 &= z.
\end{aligned}
$$

Hence

$$x\, U_1 + y\, U_2 + z\, U_3) = rE_1 + zE_3.$$

Solution 7.1

The easiest way of tackling such a question is probably to work with the attitude matrix A, say. We can check that we have a frame by showing that

$$A^tA = I,$$

and then use Theorem 7.3 to calculate the connection forms.

Here

$$A = \begin{pmatrix} \sin f/\sqrt{2} & 1/\sqrt{2} & -\cos f/\sqrt{2} \\ \sin f/\sqrt{2} & -1/\sqrt{2} & -\cos f/\sqrt{2} \\ \cos f & 0 & \sin f \end{pmatrix},$$

so

$$
\begin{aligned}
A^tA &= \begin{pmatrix} \sin f/\sqrt{2} & 1/\sqrt{2} & -\cos f/\sqrt{2} \\ \sin f/\sqrt{2} & -1/\sqrt{2} & -\cos f/\sqrt{2} \\ \cos f & 0 & \sin f \end{pmatrix} \\
&\times \begin{pmatrix} \sin f/\sqrt{2} & \sin f/\sqrt{2} & \cos f \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -\cos f/\sqrt{2} & -\cos f/\sqrt{2} & \sin f \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

after repeated use of $\sin^2 f + \cos^2 f = 1$.

In order to apply Theorem 7.3, we need to calculate the entries in dA. We use the techniques from *O'Neill*, Chapter I, for example, the chain rule gives

$$d(\sin f) = \cos f\, df.$$

Thus

$$
\begin{aligned}
\omega &= dA^tA \\
&= \begin{pmatrix} (\cos f/\sqrt{2})\, df & 0 & (\sin f/\sqrt{2})\, df \\ (\cos f/\sqrt{2})\, df & 0 & (\sin f/\sqrt{2})\, df \\ -\sin f\, df & 0 & \cos f\, df \end{pmatrix} \\
&\times \begin{pmatrix} \sin f/\sqrt{2} & \sin f/\sqrt{2} & \cos f \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -\cos f/\sqrt{2} & -\cos f/\sqrt{2} & \sin f \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & (1/\sqrt{2})\, df \\ 0 & 0 & (1/\sqrt{2})\, df \\ -(1/\sqrt{2})\, df & -(1/\sqrt{2})\, df & 0 \end{pmatrix}.
\end{aligned}
$$

Solution 7.2

Since the attitude matrix A of the natural frame field contains only constants, the derivative dA is the zero matrix. Thus

$$\omega = dA^tA = 0.$$

This result is not surprising since each vector field making up the natural frame field is parallel. As you move in any direction the natural frame field does not rotate at all so the connection forms must be identically zero.

Solution 7.3

A direct calculation shows that $A^tA = I$. However, you may have noticed a similarity with the spherical frame field. A careful study of the spherical frame field on page 83 of *O'Neill* shows that the following transforms the spherical case to the one in hand:

- interchange F_2 and F_3,
- substitute $\vartheta = f$,
- substitute $\phi = f$,
- change the sign of the new F_2.

None of these operations affect the property of being a frame.

The calculation of ω is long but not too difficult. The main step is the calculation of dA. Careful differentiation yields

$$dA = \begin{pmatrix} -2\sin f\cos f\, df & (\cos^2 f - \sin^2 f)\, df & \cos f\, df \\ (\cos^2 f - \sin^2 f)\, df & 2\sin f\cos f\, df & \sin f\, df \\ -\cos f\, df & -\sin f\, df & 0 \end{pmatrix}$$

Using this gives

$$\omega = \begin{pmatrix} 0 & -df & \cos f\, df \\ df & 0 & \sin f\, df \\ -\cos f\, df & -\sin f\, df & 0 \end{pmatrix}.$$

Solution 7.4

The attitude matrix A is

$$\begin{pmatrix} \cos\phi\cos\vartheta & \cos\phi\sin\vartheta & \sin\phi \\ -\sin\vartheta & \cos\vartheta & 0 \\ -\sin\phi\cos\vartheta & -\sin\phi\sin\vartheta & \cos\phi \end{pmatrix}.$$

Differentiating yields the following for dA.

$$\begin{pmatrix} \begin{array}{l} -\sin\phi\cos\vartheta\, d\phi \\ -\cos\phi\sin\vartheta\, d\vartheta \end{array} & \begin{array}{l} -\sin\phi\sin\vartheta\, d\phi \\ +\cos\phi\cos\vartheta\, d\vartheta \end{array} & \cos\phi\, d\phi \\ \\ -\cos\vartheta\, d\vartheta & -\sin\vartheta\, d\vartheta & 0 \\ \\ \begin{array}{l} -\cos\phi\cos\vartheta\, d\phi \\ +\sin\phi\sin\vartheta\, d\vartheta \end{array} & \begin{array}{l} -\cos\phi\sin\vartheta\, d\phi \\ -\sin\phi\cos\vartheta\, d\vartheta \end{array} & -\sin\phi\, d\phi \end{pmatrix}.$$

Use of $\omega = dA^tA$ and repeated use of $\sin^2 + \cos^2 = 1$ gives

$$\omega = \begin{pmatrix} 0 & \cos\phi\, d\vartheta & d\phi \\ -\cos\phi\, d\vartheta & 0 & \sin\phi\, d\vartheta \\ -d\phi & -\sin\phi\, d\vartheta & 0 \end{pmatrix}.$$

Solution 7.5

Linearity enables us to consider each of the terms f_iE_i separately, since

$$\nabla_V \sum_{j=1}^{3} f_jE_j = \sum_{j=1}^{3} \nabla_V f_jE_j.$$

Now

$$\begin{aligned} \nabla_V f_iE_i &= V[f_i]E_i + f_i\nabla_V E_i \quad \text{(Leibniz)} \\ &= V[f_i]E_i + f_i\sum_{j=1}^{3}\omega_{ij}(V)E_j, \\ &\qquad \text{(connection equations).} \end{aligned}$$

To obtain the result in the form given in the question involves summing the result above for $i = 1, 2, 3$ and manipulating the double sum.

$$\begin{aligned} \nabla_V W &= \sum_{i=1}^{3}\left(V[f_i]E_i + f_i\sum_{j=1}^{3}\omega_{ij}(V)E_j\right) \\ &= \sum_{i=1}^{3} V[f_i]E_i + \sum_{i=1}^{3}\sum_{j=1}^{3} f_i\omega_{ij}(V)E_j. \end{aligned}$$

Relabelling the 'summation variable' in the first sum and interchanging the order of summation in the second term gives

$$\begin{aligned} \nabla_V W &= \sum_{j=1}^{3} V[f_j]E_j + \sum_{j=1}^{3}\sum_{i=1}^{3} f_i\omega_{ij}(V)E_j \\ &= \sum_{j=1}^{3}\left(V[f_j] + \sum_{i=1}^{3} f_i\omega_{ij}(V)\right)E_j. \end{aligned}$$

Note that this result generalizes the following.

If

$$W = \sum f_iU_i$$

then

$$\nabla_V W = \sum V[f_i]U_i.$$

In the case of the natural frame field the result is simpler precisely because the connection forms for the natural frame field are identically zero.

Solution 8.1

We follow the same plan as the corresponding exercise for vector fields in the last section. Linearity allows us to deal with each term separately.

$$\begin{aligned} d(f_i\theta_i) &= df_i \wedge \theta_i + f_i d\theta_i \\ &= df_i \wedge \theta_i + f_i\sum_{j=1}^{3}\omega_{ij} \wedge \theta_j. \end{aligned}$$

Summing:

$$\begin{aligned} d\phi &= \sum_{i=1}^{3}\left(df_i \wedge \theta_i + f_i\sum_{j=1}^{3}\omega_{ij} \wedge \theta_j\right) \\ &= \sum_{i=1}^{3} df_i \wedge \theta_i + \sum_{i=1}^{3}\sum_{j=1}^{3} f_i\omega_{ij} \wedge \theta_j \\ &= \sum_{j=1}^{3} df_j \wedge \theta_j + \sum_{j=1}^{3}\sum_{i=1}^{3} f_i\omega_{ij} \wedge \theta_j \\ &= \sum_{j=1}^{3}\left(df_i + \sum_{i=1}^{3} f_i\omega_{ij}\right) \wedge \theta_j. \end{aligned}$$

Solution 8.2

(a) We follow the hint in the text, rather than that in the question. We begin by using

$$x = r\cos\vartheta,$$

$$y = r\sin\vartheta,$$

to calculate dx and dy.

$$\begin{aligned} dx &= d(r\cos\vartheta) = \cos\vartheta\, dr - r\sin\vartheta\, d\vartheta, \\ dy &= d(r\sin\vartheta) = \sin\vartheta\, dr + r\cos\vartheta\, d\vartheta. \end{aligned}$$

Let A be the attitude matrix of the cylindrical frame field, then

$$\begin{aligned} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} &= A\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \\ &= \begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\vartheta\, dr - r\sin\vartheta\, d\vartheta \\ \sin\vartheta\, dr + r\cos\vartheta\, d\vartheta \\ dz \end{pmatrix} \\ &= \begin{pmatrix} (\cos^2\vartheta + \sin^2\vartheta)dr + 0d\vartheta \\ 0dr + r(\sin^2\vartheta + \cos^2\vartheta)d\vartheta \\ dz \end{pmatrix} \\ &= \begin{pmatrix} dr \\ rd\vartheta \\ dz \end{pmatrix}. \end{aligned}$$

From this, the results follow.

(b) We apply duality and the above results in the form

$$dr = \theta_1, \quad d\vartheta = \frac{1}{r}\theta_2, \quad dz = \theta_3.$$

Now,

$$E_1[r] = dr(E_1) = \theta_1(E_1) = 1,$$

by duality.

Similarly

$$\begin{aligned} E_2[\vartheta] &= d\vartheta(E_1) \\ &= \frac{1}{r}\theta_2(E_2) \\ &= \frac{1}{r}. \end{aligned}$$

Also

$$E_3[z] = dz(E_3) = \theta_3(E_3) = 1.$$

The remaining results are obtained in exactly the same way. As a sample, we do the following.

$$\begin{aligned} E_1[\vartheta] &= d\vartheta(E_1) \\ &= \frac{1}{r}\theta_2(E_1) = 0. \end{aligned}$$

(c) We can apply the results just obtained.

First, since $V[f] = df(V)$, for any vector field V, we calculate df in terms of the dual 1-forms.

$$\begin{aligned} df &= \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \vartheta}d\vartheta + \frac{\partial f}{\partial z}dz \\ &= \frac{\partial f}{\partial r}\theta_1 + \frac{1}{r}\frac{\partial f}{\partial \vartheta}\theta_2 + \frac{\partial f}{\partial z}\theta_3. \end{aligned}$$

Applying df to E_1 and using duality gives

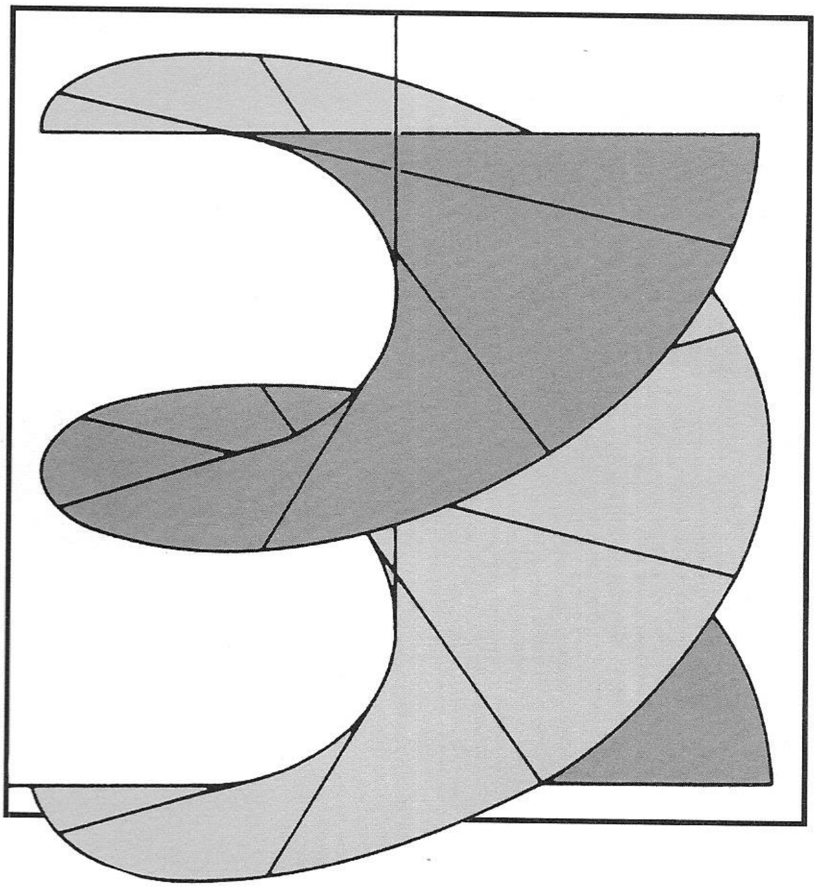
$$\begin{aligned} df(E_1) &= \left(\frac{\partial f}{\partial r}\theta_1 + \frac{1}{r}\frac{\partial f}{\partial \vartheta}\theta_2 + \frac{\partial f}{\partial z}\theta_3\right)(E_1) \\ &= \frac{\partial f}{\partial r} \times 1 + 0 + 0 \\ &= \frac{\partial f}{\partial r}. \end{aligned}$$

Similar calculations give the remaining results.

ISBN 0 7492 4771 1

M434 Part III
Mathematics: A Fourth Level Course

M434
DIFFERENTIAL GEOMETRY



PART III
EUCLIDEAN GEOMETRY

The Open University

The Open University

Mathematics: A Fourth Level Course

M434 Differential Geometry

Part III Euclidean Geometry

Prepared for the Course Team
by Bob Margolis

Set book

Barrett O'Neill, *Elementary Differential Geometry*, hardback edition (Academic Press, 1966).

It is essential to have this book; the course is based on it and will not make sense without it.

The set book is referred to as *O'Neill*.

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published as *M334* 1976.

First published as *M434* 1993. Reprinted 1996.

Copyright © 1993 The Open University

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, without written permission from the publisher or a licence from the Copyright Licensing Agency Limited. Details of such licences (for reprographic reproduction) may be obtained from the Copyright Licensing Agency Ltd of 90 Tottenham Court Road, London, W1P 9HE.

Edited, designed and typeset by the Open University using the Open University TEX System.

Printed in the United Kingdom by Hobbs the Printers Ltd, Totton, Hampshire.

ISBN 0 7492 4772 X

This text forms part of an Open University Fourth Level Course. If you would like a copy of *Studying with The Open University*, please write to the Central Enquiry Service, PO Box 200, The Open University, Walton Hall, Milton Keynes, MK7 6YZ. If you have not already enrolled on the Course and would like to buy this or other Open University material, please write to Open University Educational Enterprises Ltd, 12 Cofferidge Close, Stony Stratford, Milton Keynes, MK11 1BY, United Kingdom.

1.2

Contents

Introduction 4

1 Isometries 5

2 Derivative maps of isometries 7

3 Orientation 9

4 Euclidean geometry 10

5 Congruence of curves 12

6 Summary 15

Solutions to the exercises 16

Introduction

This part rounds off the study of curves as such, although curves will play an important role in the study of surfaces. The main aim of Chapter III of *O'Neill* is to prove a theorem to the effect that the curvature and torsion functions of a unit-speed curve completely determine the curve, except for the precise position of the route of the curve in $\mathbf{E}^3$.

In order to state the theorem precisely, we have to define exactly what is meant by two curves being 'the same'. The notion that we need is that of an **isometry** of $\mathbf{E}^3$.

We shall actually define what we mean by the routes being the same.

The first four sections are devoted to isometries. Section 1 defines isometries, links them to ideas of linear algebra and then shows that all isometries can be expressed in a form that makes computations easy.

Section 2 investigates the derivative maps of isometries. Since we shall be dealing with curves related by an isometry, we need to understand how isometries act on tangent vectors defined on the curves; this is precisely what the derivative map tells us.

In Section 3 we tackle the problem of distinguishing between left-handed and right-handed objects. The concept that we require is **orientation**. The need for such a concept can be illustrated by considering a circular helix, say, and its reflection in the xy-plane plane. In all respects except the 'handedness' of the spiral, the two helices are the same. The process of reflection (an isometry) is different in some essential respect from, say, translation (another isometry).

With the ideas from Sections 1–3, it remains to find out only how isometries affect acceleration of curves. We know that, in general, mappings do *not* preserve acceleration. However, isometries do and this will be important for the theorem that we are aiming to prove.

This was tackled in Chapter I.

Finally, Section 5 puts together the ideas about curves embodied in the Frenet apparatus and formulas with the work on isometries.

It may not always be apparent where the detailed work on isometries is leading. It may help if you remember that all the information about a curve α is contained in α and its first three derivatives. So, to understand the relationship between curves and isometries, we must be able to describe isometries and to understand what isometries (or, to be precise, their derivative maps) do to the derivatives of a curve.

We have formulas for the Frenet apparatus in terms of α and its first three derivatives.

Study advice

The following represents a possible plan for study weeks.

Week 1 *O'Neill*, Chapter III, Sections 1–3.

Week 2 *O'Neill*, Chapter III, Sections 4–5, the summary and TMA02.

1 Isometries

Read O'Neill: Chapter III, Section 1, pages 98–102.

Erratum *O'Neill*, page 102, the last line of text should read:

'Alternatively, ..., $q = F(p) \ldots$'. ■

This section has two aims.

(a) To define what is meant by an isometry.

(b) To show that any isometry can be expressed as an orthogonal transformation followed by a translation.

The second aim is important because it makes calculations with isometries very easy. It also makes the construction of an isometry to do a particular task a relatively straightforward process.

Isometries The mathematical concept of isometry is intended to formalize the idea of a *rigid motion*, a motion which changes the position of objects but not their size or shape. In particular, the distance between two points after the rigid motion must be the same as the distance between them before it. This property of preserving distance is taken as the defining property of isometries.

Many of the transformations that you have met before are isometries. Intuitively, it is fairly clear that rotations, reflections and translations are isometries. *O'Neill* deals explicitly with translations and rotations in his examples. The 'straightforward computation' referred to in the discussion of rotations is elementary, but messy, algebra. The result is the important thing, not the details of the justification.

M101 and *M203*

Composites and inverses Having defined a new type of object, isometries, you should be expecting the next steps. We show that the composite of isometries is also an isometry and that the inverse of an isometry is also an isometry.

The set of isometries is closed under composition and taking inverses.

Actually, *O'Neill* deals only with composition (in Lemma 1.3). *O'Neill* also observes that the inverse of a translation is a translation and, therefore, an isometry. The general result for inverses is not difficult and we include the proof here for completeness.

The idea of the proof is straightforward: because an isometry F preserves the distance between any two points of $\mathbf{E}^3$, in particular it will preserve the distance between $F^{-1}(\mathrm{p})$ and $F^{-1}(\mathrm{q})$.

Lemma 1.3a If F is an isometry of $\mathbf{E}^3$, then F^{-1} is also an isometry of $\mathbf{E}^3$.

Proof Suppose that p and q are points in $\mathbf{E}^3$. Then,

$$\begin{aligned} d(F^{-1}(\mathrm{p}), F^{-1}(\mathrm{q})) &= d(F(F^{-1}(\mathrm{p})), F(F^{-1}(\mathrm{q}))) \quad \text{(because } F \text{ is an isometry)} \\ &= d(\mathrm{p}, \mathrm{q}) \quad \text{(because } FF^{-1} \text{ is the identity).} \end{aligned}$$

Thus

$$d(F^{-1}(\mathrm{p}), F^{-1}(\mathrm{q})) = d(\mathrm{p}, \mathrm{q}),$$

and F^{-1} is an isometry.

Orthogonal transformation *O'Neill* defines this, in passing, as a linear transformation which preserves dot products. Since angles are, essentially, defined in terms of dot products by

$$\begin{aligned} \cos\theta &= \frac{\mathrm{p}\cdot\mathrm{q}}{\|\mathrm{p}\|\,\|\mathrm{q}\|} \\ &= \frac{\mathrm{p}\cdot\mathrm{q}}{\sqrt{\mathrm{p}\cdot\mathrm{p}}\sqrt{\mathrm{q}\cdot\mathrm{q}}}, \end{aligned}$$

orthogonal transformations preserve angles as well as dot products. In particular, they preserve right angles, hence the name *orthogonal.*

The plan of campaign for the proof of Theorem 1.7 can be set out as follows.

(a) Deal with isometries that fix the origin by;

  (i) showing that such an isometry preserves dot products;

  (ii) showing such an isometry is linear;

  (iii) deducing that such an isometry is an orthogonal transformation.

(a) Deal with the general case by combining it with a translation that ensures that the composite fixes the origin and, hence, that the composite is an orthogonal transformation.

Lemma 1.6 Two points arise here: polarization and linearity.

Polarization is the sort of idea that is 'obvious' once you have seen it! Actually, it simply involves careful application of the information about F and the definitions of norm and distance in terms of dot product.

The final steps in the linearity argument are omitted by *O'Neill* for the very good reason that they take up a lot of space for little insight. However, for completeness, here is the missing detail.

Suppose that

$$\begin{aligned}\mathbf{p} &= (p_1, p_2, p_3),\\ \mathbf{q} &= (q_1, q_2, q_3),\\ F(\mathbf{u}_i) &= \mathbf{v}_i, \quad i = 1, 2, 3.\end{aligned}$$

Then

$$\begin{aligned}F(a\mathbf{p} + b\mathbf{q}) &= F\left(\sum_{i=1}^{3}(ap_i\mathbf{u}_i + bq_i\mathbf{u}_i)\right)\\ &= F\left(\sum_{i=1}^{3}(ap_i + bq_i)\mathbf{u}_i\right)\\ &= \sum_{i=1}^{3}(ap_i + bq_i)F(\mathbf{u}_i)\\ &\quad\text{(because we know that } F \text{ acts linearly on the } \mathbf{u}_i)\\ &= \sum_{i=1}^{3}(ap_i + bq_i)\mathbf{v}_i\\ &= \sum_{i=1}^{3}(ap_i\mathbf{v}_i + bq_i\mathbf{v}_i)\\ &= a\sum_{i=1}^{3} p_i\mathbf{v}_i + b\sum_{i=1}^{3} q_i\mathbf{v}_i\\ &= aF(\mathbf{p}) + bF(\mathbf{q}).\end{aligned}$$

Once you know that F behaves linearly with respect to a basis (the $\mathbf{u}_i$) then it must behave linearly overall.

In the next section we shall apply the ideas contained in the proof of Lemma 1.6 to obtain an explicit formula for the orthogonal transformation that takes one given orthonormal basis into another.

Optional The comments about closure and inverses, made earlier, may have reminded you of work on groups from previous courses. If time permits, you might like to look at Exercises 7 and 8 which discuss groups of isometries.

Exercise 1.1 *O'Neill*, page 103, Exercise 1. (This exercise is included to emphasize that the order of the orthogonal part and translation part of an isometry is important.)

Exercise 1.2 *O'Neill*, page 103, Exercise 2. (*Hint*: Apply the first exercise.)

Exercise 1.3 *O'Neill*, page 103, Exercise 4.

Exercise 1.4 *O'Neill*, page 103, Exercise 5.

Exercise 1.5 *O'Neill*, page 103, Exercise 6.

[*Solutions on page 16*]

2 Derivative maps of isometries

Read *O'Neill: Chapter III, Section 2, pages 104–106.*

Having obtained the computationally useful expression

$$F = TC$$

for an isometry, we now go on to consider the derivative map of an isometry. In a way, the result of Theorem 2.1 is almost inevitable. The orthogonal part of an isometry determines (in a crude sense) how geometric objects will be 'rotated' whereas the translation part alters only their position in space. Because an isometry is a rigid motion, any collection of tangent vectors attached to an object will all be 'rotated' in the same way. Expressed precisely, that is exactly what Theorem 2.1 says.

In matrix form, the vector part of $F_*(\mathbf{v_p})$ is given by

$$C\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Corollary 2.2 There is a further deduction that we can make. By applying the pointwise principle, we can say that

$$F_*(V) = C(V)$$

for any vector field on $\mathbf{E}^3$. Because F_* preserves dot products, we can deduce that

$$F_*(V) \cdot F_*(W) = V \cdot W$$

for any vector fields V and W on $\mathbf{E}^3$. It also follows that F_* maps frame fields to frame fields.

If E_1, E_2 and E_3 form a frame field, then

$$\begin{aligned} F_*(E_i) \cdot F_*(E_j) &= E_i \cdot E_j \\ &= \delta_{ij}. \end{aligned}$$

Hence $F_*(E_1)$, $F_*(E_2)$ and $F_*(E_3)$ form a frame field.

Theorem 2.3 With this theorem and the remark that follows, the main aim of this section has been achieved.

Given two points, each equipped with a frame, we can find the unique isometry that maps the first point-and-frame to the second.

Strictly, the isometry maps the point, its derivative map deals with the frame.

It may help to understand where the expression

$$C = {}^tB\,A$$

comes from if we look in a little more detail at exactly what is required of C.

The notation set up in Theorem 2.3 means that

$$\begin{aligned}
e_1 &= (a_{11}, a_{12}, a_{13}) \\
e_2 &= (a_{21}, a_{22}, a_{23}) \\
e_3 &= (a_{31}, a_{32}, a_{33}) \\
f_1 &= (b_{11}, b_{12}, b_{13}) \\
f_2 &= (b_{21}, b_{22}, b_{23}) \\
f_3 &= (b_{31}, b_{32}, b_{33}).
\end{aligned}$$

We want $C(e_i) = f_i$ so we want C to have the following effect.

$$\begin{aligned}
C\begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} &= \begin{pmatrix} b_{11} \\ b_{12} \\ b_{13} \end{pmatrix} \\
C\begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix} &= \begin{pmatrix} b_{21} \\ b_{22} \\ b_{23} \end{pmatrix} \\
C\begin{pmatrix} a_{31} \\ a_{32} \\ a_{33} \end{pmatrix} &= \begin{pmatrix} b_{31} \\ b_{32} \\ b_{33} \end{pmatrix}.
\end{aligned}$$

If we combine these as a single matrix equation, we get

$$C\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}.$$

We can write this equation as

$$C\,{}^tA = {}^tB.$$

We can solve this for C by multiplying on the right by $({}^tA)^{-1}$ and, since A and ${}^tA$ are orthogonal, $({}^tA)^{-1} = {}^t({}^tA) = A$. Hence

$$\begin{aligned}
C &= {}^tB\,{}^tA^{-1} \\
&= {}^tB\,A.
\end{aligned}$$

A point easily overlooked is that the translation part of the isometry is

$$\mathbf{q} - C(\mathbf{p})$$

and *not* $\mathbf{q} - \mathbf{p}$. This is a consequence of the expression

$$F = TC$$

in which the orthogonal part is applied first. Overall, we require

$$F(\mathbf{p}) = \mathbf{q}$$

but

$$\begin{aligned}
F(\mathbf{p}) &= (TC)(\mathbf{p}) \\
&= T(C(\mathbf{p})).
\end{aligned}$$

Thus T is the translation taking $C(\mathbf{p})$ to $\mathbf{q}$.

We can now predict that, when an isometry is applied to a curve, the orthogonal part will determine what happens to the Frenet frame at each point of the curve.

In spite of the fact that applying Theorem 2.3 is a very important skill, we have set only one exercise on it. The reason is that the application to curves will give you further opportunity to develop this skill.

Exercise 2.1 *O'Neill*, page 106, Exercise 1.

Exercise 2.2 *O'Neill*, page 107, Exercise 5.

[*Solutions on page 18*]

3 Orientation

Read *O'Neill: Chapter III, Section 3, pages 107–110.*

Errata

1 *O'Neill*, page 108, the second displayed equation in the proof of Lemma 3.2 should read:

$$\left(\sum_k c_{jk} a_{ik}\right) = \left(\sum_k a_{ik} {}^t c_{kj}\right) = A\,{}^tC.$$

2 *O'Neill*, page 108, the final calculations in the proof of Lemma 3.2 should read:

$$\begin{aligned} F_*(\mathrm{e}_1) \cdot F_*(\mathrm{e}_2) \times F_*(\mathrm{e}_3) &= \det(A\,{}^tC) \\ &= \det A \det {}^tC \\ &= \det A \det C \\ &= \operatorname{sgn} F \mathrm{e}_1 \cdot \mathrm{e}_2 \times \mathrm{e}_3. \end{aligned}$$

■

Orientation The underlying idea behind the formal definition of orientation (left- and right-handed gloves) is probably not as elusive as *O'Neill* suggests. It is the formalization that is tricky.

Lemma 3.2 You may find the following, expanded, version of the proof helpful.

Let us denote the attitude matrix of the frame field $F_*(\mathrm{e}_i)$, $i = 1, 2, 3$ by B. Then, writing out the effect of F_*, that is C, in full, we get

$$\begin{pmatrix} b_{11} \\ b_{12} \\ b_{13} \end{pmatrix} = C \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix}$$

$$\begin{pmatrix} b_{21} \\ b_{22} \\ b_{23} \end{pmatrix} = C \begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix}$$

$$\begin{pmatrix} b_{31} \\ b_{32} \\ b_{33} \end{pmatrix} = C \begin{pmatrix} a_{31} \\ a_{32} \\ a_{33} \end{pmatrix}.$$

Combining these, we have

$${}^tB = C\,{}^tA.$$

Now, the triple scalar product

$$F_*(\mathrm{e}_1) \cdot F_*(\mathrm{e}_2) \times F_*(\mathrm{e}_3)$$

is the determinant of the attitude matrix B. But,

$$\begin{aligned} \det(B) &= \det({}^t(C\,{}^tA)) \\ &= \det({}^{tt}A\,{}^tC) \\ &= \det(A\,{}^tC) \\ &= \det(A)\ \det({}^tC) \\ &= \det(A)\ \det(C) \\ &= \operatorname{sgn} F\, \mathrm{e}_1 \cdot \mathrm{e}_2 \times \mathrm{e}_3. \end{aligned}$$

Exercise 3.1 On page 107, Exercise 5 of *O'Neill*, two frames are defined. Find their respective orientations.

In the last section you found an isometry, F, taking the first frame to the second. Find sgn F and hence verify Lemma 3.2 for this example.

Exercise 3.2 Let the orthogonal transformation C be specified by the matrix

$$\begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

and let the vectors $\mathbf{v}$ and $\mathbf{w}$ be given by

$$\mathbf{v} = (3, 1, -1) \quad \text{and} \quad \mathbf{w} = (-3, -3, 1).$$

Check the formula

$$C(\mathbf{v} \times \mathbf{w}) = \operatorname{sgn} C(\mathbf{v}) \times C(\mathbf{w}).$$

[*Solutions on page 18*]

4 *Euclidean geometry*

Read *O'Neill: Chapter III, Section 4, pages 112–115.*

Erratum *O'Neill*, page 113, in Fig. 3.6, the vector attached to $\overline{\alpha}(t)$ should be labelled $\overline{Y}(t)$, *not* $Y(t)$. ■

This section provides the final technical tools that we shall need for our discussion of congruence of curves and the first half of the theorem that is the main aim of this part of the course.

The central theme of this section is finding out what happens when an isometry is applied to a curve. We already know how to find the effect of the isometry on the Frenet frame that is attached to each point on the curve: we apply the derivative map. For isometries, the derivative map has a particularly simple form. We also need to know about the relationship between the Frenet formulas for the two curves and these involve the *derivatives* of the Frenet frame. The main result of this section describes exactly how isometries affect derivatives of vector fields. This result is then applied to the Frenet frame of a unit-speed curve.

We know this from the general work on mappings.

Acceleration and mappings You might like to work through the compressed example in the middle of page 112 in a little more detail. We suggest that you ignore the remark about diffeomorphisms.

Corollary 4.1 It is not *quite* clear of what this is a corollary! Actually, it follows from the fact that the derivative map of an isometry is effected by the *same* matrix at each point of $\mathbf{E}^3$. The core of the proof is the paragraph beginning: 'On the other hand...'

Theorem 4.2 This is the application of Corollary 4.1 to curves.

In *Part II*, Section 4, we showed that all the Frenet apparatus of a curve α can be expressed in terms of the first three derivatives of α by using dot products, norms and cross products. These derivatives define three vector fields on the curve and Corollary 4.1 tells us that such vector fields are preserved when we apply an isometry to the curve. Isometries also preserve dot products, norms and (making allowances for sign) cross products.

Taking these facts into account, it is reasonable to suppose that applying an isometry to a curve gives a new curve with, essentially, the same Frenet apparatus as the old one. The theorem makes these ideas precise and then proves that the expected actually happens.

The proof makes use of some deductions from the definitions of the Frenet apparatus. Since β is unit speed, we have

$$T = \beta',$$
$$\kappa = \|T'\| = \|\beta''\|,$$
$$N = \frac{T'}{\kappa} = \frac{\beta''}{\kappa}.$$

When showing that $\overline{\kappa} = \kappa$, the proof uses the fact that F_* preserves dot products, and hence norms, in the line

$$\ldots \|F_*(\beta'')\| = \|\beta''\| \ldots.$$

In the calculation of $\overline{N}$, the proof uses the linearity of F_* by asserting that

$$\frac{F_*(\beta'')}{\kappa} = F_*\left(\frac{\beta''}{\kappa}\right).$$

Note, also, that *O'Neill* uses the modified definition of τ that we gave in *Part II*, Section 3.

That is $\tau = -B' \cdot N$.

Anticipating the beginning of the next section a little, we define two curves α and β to be congruent if there is an isometry F such that

$$F(\alpha) = \beta.$$

This theorem can then be restated in the following form.

If α and β are congruent unit-speed curves, then

$$\kappa_\alpha = \kappa_\beta \quad \text{and} \quad \tau_\alpha = \pm\tau_\beta.$$

The sign in the second equation is determined by the sign of the isometry which makes the two curves congruent.

The next section proves the converse: that if we know that two unit-speed curves have the same curvature and (making the allowance for sign) the same torsion, then the curves must be congruent. We shall show explicitly also how to produce the isometry to justify this assertion.

The two results together show that the curvature and torsion give a complete geometric description of a curve.

Non unit-speed curves The results in *O'Neill* on the effect of isometries on curves deal only with unit-speed curves. It is useful to add one further result.

Lemma 4.4 If α is a curve in $\mathbf{E}^3$ and F is an isometry, then

$$v_{F(\alpha)} = v_\alpha,$$

that is, F preserves speed.

Proof We know that F is a mapping and so preserves velocity, that is,

$$(F(\alpha))' = F_*(\alpha').$$

Now

$$\begin{aligned} v_{F(\alpha)} &= \|\,(F(\alpha))'\,\| \\ &= \|F_*(\alpha')\| \\ &= \sqrt{F_*(\alpha') \cdot F_*(\alpha')} \\ &= \sqrt{\alpha' \cdot \alpha'} \quad \text{(because isometries preserve dot products)} \\ &= \|\alpha'\| \\ &= v_\alpha. \end{aligned}$$

This completes the proof.

Exercise 4.1 *O'Neill*, page 115, Exercise 1.

Exercise 4.2 *O'Neill*, page 115, Exercise 2.

[*Solutions on page 19*]

5 Congruence of curves

Read *O'Neill: Chapter III, Section 5, pages 116–121.*

Erratum *O'Neill*, page 120, the last line of the proof of Corollary 5.6 should read:

$$F(\alpha(t)) = F(\overline{\alpha}(s(t))) = \overline{\beta}(s(t)) = \beta(t). \qquad \blacksquare$$

In this section we prove the converse of Theorem 4.2 to complete the proof of the following result.

Main theorem on curves If α and β are unit speed curves in $\mathbf{E}^3$ then α and β are congruent if, and only if,

$$\kappa_\beta = \kappa_\alpha \quad \text{and} \quad \tau_\beta = \pm\tau_\alpha.$$

A by-product of the method of proof is that it is possible to give an explicit recipe for constructing the isometry in such cases.

We also extend the theorem to non unit-speed curves. This extension requires the additional condition that

$$v_\beta = v_\alpha > 0.$$

The overall strategy is the following.

(a) Deal with the case in which the two curves are related by a translation.

(b) Deal with the general case, making use of the special case.

Let us consider two curves, one a translated version of the other. We know that, for a translation, the orthogonal part, C is represented by the identity matrix. Since it is C that determines what happens to the velocity vector of a curve, velocity vectors of the two curves will actually have the *same* Euclidean coordinate functions. Thus the velocity vectors will be parallel.

As usual, when discussing isometries, we use the $F = TC$ notation without comment.

Lemma 5.2 This lemma deals with the converse of the observation above. It shows that we can detect if two given curves are related by a translation by inspecting their velocities. If their velocity vector fields are parallel, there is a translation mapping one curve to the other.

Note that you cannot actually calculate the three constants $p_i, i = 1, 2, 3$ unless you know a pair of corresponding points on the two curves. Without this knowledge all that you can assert is the *existence* of the constants. Fortunately, this is sufficient for our purposes.

Theorem 5.3 The approach to the general case is mildly complicated by the need to consider the cases

$$\tau_\beta = \tau_\alpha \quad \text{and} \quad \tau_\beta = -\tau_\alpha$$

separately.

The basic idea is fairly straightforward. We choose corresponding points on each curve and choose an isometry that will accomplish the following:

- map $\alpha(0)$ to $\beta(0)$,
- map the Frenet frame at $\alpha(0)$ to the Frenet frame at $\beta(0)$, allowing for a small complication.

Because equality of the *functions* κ_α and κ_β implies that

$$\kappa_\alpha(0) = \kappa_\beta(0),$$

we choose $\alpha(0)$ and $\beta(0)$ as the pair of corresponding points.

We can always construct an isometry to map a given point to another and, simultaneously, map a chosen frame at the first point to a frame at the second.

This was the technique you practised in Section 2.

The 'small complication' referred to above is the fact that Frenet frames are, by definition, always positively oriented. We know that we need to consider orientation reversing isometries and so the isometry which simply maps one Frenet frame to the other will not always be quite what is needed..

We construct the isometry to map T_α and N_α to T_β and N_β respectively. We then arrange

$$\begin{aligned} B_\alpha &\longmapsto B_\beta \quad \text{if } \tau_\alpha = \tau_\beta, \\ B_\alpha &\longmapsto -B_\beta \quad \text{if } \tau_\alpha = -\tau_\beta. \end{aligned}$$

Intuitively, the isometry F, constructed to this recipe, ensures that $\overline{\alpha} = F(\alpha)$ and β coincide at $\beta(0)$ *and* set off in the same direction from $\beta(0)$. The final stage in the proof is to show that, not only do they set off in the same direction but they remain exactly the same. To do so, we show that their velocities are the same. This implies that they are related by a translation. However, the two curves coincide at $t = 0$, so the translation is the identity. That is an involved way of saying that

This is why we needed the special case first.

$$\overline{\alpha} = F(\alpha) = \beta.$$

The detail of the proof that the tangents are parallel,

$$\overline{T} = T$$

in the notation of the proof, gets a little involved. Let us look carefully at what *O'Neill* does.

The expression

$$\overline{T} \cdot T + \overline{N} \cdot N + \overline{B} \cdot B$$

contains three terms, each of which is the dot product of two unit vectors. Using the Schwarz inequality is rather a case of overkill. If $\overline{T}$ and T are at an angle θ then we know that

$$\begin{aligned} \overline{T} \cdot T &= \|\overline{T}\|\,\|T\| \cos\theta \\ &\leq \|\overline{T}\|\,\|T\| \quad (\text{because } -1 \leq \cos\theta \leq +1) \\ &= 1 \quad (\text{both are unit vectors}). \end{aligned}$$

Equality will occur only when $\cos\theta = 1$ and the vectors are equal.

Two similar results hold for the other two terms and we deduce that

$$\overline{T} \cdot T + \overline{N} \cdot N + \overline{B} \cdot B \leq 3$$

with equality only when corresponding vectors are equal.

Having established that the function

$$f = \overline{T} \cdot T + \overline{N} \cdot N + \overline{B} \cdot B$$

has the value 3 at one point, namely $t = 0$, we go on to show that its derivative is zero, so that f is constant. It is at this point that the Frenet formulas are used. The details omitted by *O'Neill* are as follows.

$$\begin{aligned} f' &= \overline{T}' \cdot T + \overline{T} \cdot T' + \overline{N}' \cdot N + \overline{N} \cdot N' + \overline{B}' \cdot B + \overline{B} \cdot B' \\ &= \overline{\kappa}\overline{N} \cdot T + \overline{T} \cdot \kappa N + (-\overline{\kappa}\overline{T} + \overline{\tau}\overline{B}) \cdot T + \overline{N} \cdot (-\kappa T + \tau B) - \overline{\tau}\overline{N} \cdot B - \overline{B} \cdot \tau N \\ &= 0. \end{aligned}$$

It should be clear that the terms *do* cancel in pairs as *O'Neill* asserts.

Comment: You may well have been unhappy about writing down expressions such as

If you did not spot this inconsistency, do not worry! The comment is just in case you did.

$$\overline{T} \cdot T$$

before we have proved that the two curves are the same. Until the proof is complete, we really cannot be sure that $\overline{T}$ and T have the same point of application. You are intended to interpret these expressions as the result of taking the dot products after moving both vectors to the origin.

Example 5.4 Here *O'Neill* points out the application of the method of Section 2 to finding the isometry required to map one curve to another.

Corollary 5.5 We already have a test for deciding whether or not a curve is a *cylindrical* helix: calculate the ratio τ/κ. The curve is a cylindrical helix if, and only if, this ratio is a constant.

What *O'Neill* is discussing here are *circular* helices.

Corollary 5.6 This corollary removes the restriction to unit-speed curves.

Theorem 5.7 *O'Neill* presents this for completeness and because of the application to which he refers. Since the part of *O'Neill* referred to does not form part of this course, you may omit this theorem, or simply read it out of interest.

The achievements of this section are substantial. The Frenet apparatus gives a standard test for the congruence of curves. Moreover, if two curves are found to be congruent, then we have a method of constructing the isometry which makes them congruent.

In constructing such isometries, the only point that needs to be watched is what happens to B. Symbolically

$$\left.\begin{array}{c} T_\alpha(0) \\ N_\alpha(0) \\ B_\alpha(0) \end{array}\right\} \longmapsto \left\{\begin{array}{c} T_\beta(0) \\ N_\beta(0) \\ \pm B_\beta(0) \end{array}\right.$$

with the sign chosen to match the sign in $\tau_\beta = \pm\tau_\alpha$.

The first two exercises give you a chance to practise the basic techniques resulting from Theorem 5.3. The others develop the ideas slightly and may be omitted if you are short of time.

Exercise 5.1 The curves α and β are defined by

$$\alpha(t) = (\cos t, \sin t, 0),$$
$$\beta(t) = \tfrac{1}{3}(-2\cos t + 2\sin t + 3, 2\cos t + \sin t - 3, \cos t + 2\sin t).$$

Show that α and β are congruent and find an isometry F such that

$$\beta = F(\alpha).$$

Exercise 5.2 The curves α and β are defined by

$$\alpha(t) = (3t - t^3, 3t^2, 3t + t^3),$$
$$\beta(t) = (3\sqrt{2}t, -3t^2, \sqrt{2}t^3).$$

Show that α and β are congruent and find an isometry F such that

$$\beta = F(\alpha).$$

(*Hint*: To save you some time, note that α was discussed in Chapter II, Example 4.4 on pages 69–70 of *O'Neill*. Use the results obtained there to shorten your working.)

Exercise 5.3 *O'Neill*, page 121, Exercise 1.

Exercise 5.4 *O'Neill*, page 122, Exercise 4.

[*Solutions on page 20*]

6 Summary

Read *O'Neill: Chapter III, Section 6, page 123.*

This chapter has introduced the formal definitions necessary to give a definition of congruence of curves in $\mathbf{E}^3$.

Using these ideas it was possible to prove that the speed, curvature and torsion functions completely determine a curve in $\mathbf{E}^3$, except for the position of its route in space.

The techniques which have been developed from the theory are:

(a) deciding whether or not two curves are congruent;

(b) when two curves *are* congruent, constructing an isometry that maps one to the other.

This part of the course completes the direct study of curves. In the next part we begin the study of surfaces; some of our results about curves will be useful tools in this study.

Solutions to the exercises

Solution 1.1

In order to prove that two transformations (i.e. functions) are the same, we show that they have the same effect on a typical member of the domain. So, let $\mathbf{p} \in \mathbf{E}^3$. Then

$$\begin{aligned}(CT_a)(\mathbf{p}) &= C(T_a(\mathbf{p}))\\ &= C(\mathbf{p}+\mathbf{a}) \quad \text{(definition of } T_a\text{)}\\ &= C(\mathbf{p}) + C(\mathbf{a}) \quad (C \text{ is linear})\\ &= T_{C(a)}(C(\mathbf{p})) \quad \text{(adding } C(a) \text{ is } T_{C(a)}\text{)}\\ &= (T_{C(a)}C)(\mathbf{p}).\end{aligned}$$

Hence

$$CT_a = T_{C(a)}C.$$

Solution 1.2

As suggested in the hint, we apply the above result.

$$\begin{aligned}FG &= (T_aA)(T_bB)\\ &= T_a(AT_b)B\\ &= T_a(T_{A(b)}A)B \quad \text{(using the previous solution)}\\ &= (T_aT_{A(b)})(AB)\\ &= T_{a+A(b)}(AB).\end{aligned}$$

Hence the translation part of FG is

$$T_{a+A(b)},$$

and the orthogonal part is

$$AB,$$

the composite of the orthogonal parts of F and of G.

Note that the translation part is not simply the composite of the separate translation parts.

To deal with GF, we could repeat the entire calculation. However, it is easier just to interchange a with b, A with B. Thus, the translation part of GF is

$$T_{b+B(a)}$$

and the orthogonal part is

$$BA.$$

Solution 1.3

The most straightforward check that C is orthogonal is to calculate ${}^tC\,C$.

$$\begin{aligned}{}^tC\,C &= \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}\\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.\end{aligned}$$

Now, in matrix form,

$$\begin{aligned}C(\mathbf{p}) &= \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ -6 \end{pmatrix}\\ &= \begin{pmatrix} \frac{2}{3} \\ \frac{19}{3} \\ -\frac{7}{3} \end{pmatrix}.\end{aligned}$$

$$\begin{aligned}C(\mathbf{q}) &= \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}\\ &= \begin{pmatrix} -\frac{5}{3} \\ -\frac{4}{3} \\ \frac{7}{3} \end{pmatrix}.\end{aligned}$$

Now,

$$\begin{aligned}C(\mathbf{p})\cdot C(\mathbf{q}) &= -\frac{2}{3}\frac{5}{3} - \frac{19}{3}\frac{4}{3} - \frac{7}{3}\frac{7}{3}\\ &= -\frac{135}{9}\\ &= -15.\\ \mathbf{p}\cdot\mathbf{q} &= 3\times 1 + 1\times 0 - 6\times 3\\ &= -15\\ &= C(\mathbf{p})\cdot C(\mathbf{q}).\end{aligned}$$

Solution 1.4

(a) Since

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 8 \end{pmatrix} = \begin{pmatrix} -3\sqrt{2} \\ -2 \\ 5\sqrt{2} \end{pmatrix},$$

we have

$$C(\mathbf{p}) = (-3\sqrt{2}, -2, 5\sqrt{2}).$$

Hence

$$\begin{aligned}\mathbf{q} &= F(\mathbf{p})\\ &= (T_aC)(\mathbf{p})\\ &= T_a((-3\sqrt{2}, -2, 5\sqrt{2}))\\ &= (-3\sqrt{2}, -2, 5\sqrt{2}) + (1, 3, -1)\\ &= (-3\sqrt{2}+1, 1, 5\sqrt{2}-1).\end{aligned}$$

(b) For this part, we can usefully apply the solution to the first exercise to get an explicit expression for F^{-1}.

$$\begin{aligned}F^{-1} &= (T_aC)^{-1}\\ &= C^{-1}T_a^{-1}\\ &= {}^tC\,T_{-a}\\ &= T_{{}^tC(-a)}\,{}^tC.\end{aligned}$$

We now calculate the image of $\mathbf{p}$ in exactly the same way as above. Firstly,

$$
{}^tC(\mathbf{p}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 8 \end{pmatrix}
$$

$$
= \begin{pmatrix} 5\sqrt{2} \\ -2 \\ 3\sqrt{2} \end{pmatrix},
$$

we have

$$
{}^tC(\mathbf{p}) = (5\sqrt{2}, -2, 3\sqrt{2}).
$$

Now

$$
-\mathbf{a} = (-1, -3, 1),
$$

so

$$
{}^tC(-\mathbf{a}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 \\ -3 \\ \sqrt{2} \end{pmatrix}.
$$

Thus

$$
\begin{aligned}
\mathbf{q} &= F^{-1}(\mathbf{p}) \\
&= (5\sqrt{2}, -5, 4\sqrt{2}).
\end{aligned}
$$

(c) We have

$$
T_a(\mathbf{p}) = (3, 1, 7)
$$

and

$$
\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} -2\sqrt{2} \\ 1 \\ 5\sqrt{2} \end{pmatrix}.
$$

Thus

$$
\mathbf{q} = (CT_a)(\mathbf{p}) = (-2\sqrt{2}, 1, 5\sqrt{2}).
$$

Solution 1.5

There are several possible approaches to this question. Perhaps the most obvious is to begin by using the definition of isometry as a test: check if

$$
\|F(\mathbf{p}) - F(\mathbf{q})\| = \|\mathbf{p} - \mathbf{q}\|.
$$

Alternatively, one could try to express each transformation in the form

$$
T_aC,
$$

where C is *some* matrix transformation and then check C for orthogonality.

On the whole we shall adopt the 'first principles' approach. The problem with this is that if F is not an isometry, it is sometimes difficult to see why not.

(a) Here we have

$$
\begin{aligned}
\|F(\mathbf{p}) - F(\mathbf{q})\| &= \| -\mathbf{p} - (-\mathbf{q})\| \\
&= \|\mathbf{q} - \mathbf{p}\| \\
&= \|\mathbf{p} - \mathbf{q}\|.
\end{aligned}
$$

Thus F is an isometry.

(b) In this case

$$
\begin{aligned}
\|F(\mathbf{p}) - F(\mathbf{q})\| &= \|\mathbf{p} \cdot \mathbf{a}\,\mathbf{a} - \mathbf{q} \cdot \mathbf{a}\,\mathbf{a}\| \\
&= \|(\mathbf{p} - \mathbf{q}) \cdot \mathbf{a}\,\mathbf{a}\| \\
&= \|\mathbf{p} - \mathbf{q}\|\,\|\mathbf{a}\| \cos\theta\,\|\mathbf{a}\| \\
&\qquad \text{(where } \theta \text{ is the angle between } \mathbf{p} - \mathbf{q} \text{ and } \mathbf{a} \\
&= \|\mathbf{p} - \mathbf{q}\| \cos\theta \quad (\text{since } \|\mathbf{a}\| = 1).
\end{aligned}
$$

Since there is no reason why θ should be zero for all $\mathbf{p}$ and $\mathbf{q}$, F is not an isometry. This observation enables us to construct a counterexample; all we need to do is choose $\mathbf{p}$ and $\mathbf{q}$ so that $\theta \neq 0$. To do so, let $\mathbf{p}$ be any non-zero vector perpendicular to $\mathbf{a}$ and let $\mathbf{q}$ be the zero vector. Then $\mathbf{p} - \mathbf{q}$ is orthogonal to $\mathbf{a}$ and so $\cos\theta = 0$. Hence

$$
\|\mathbf{p}\| \neq 0
$$

but

$$
\|F(\mathbf{p})\| = \|\mathbf{0}\| = 0.
$$

(c) If we write

$$
F(\mathbf{p}) = (p_3, p_2, p_1) - (1, 2, 3),
$$

then we can see that $F = TC$, where T is translation by

$$
(-1, -2, -3),
$$

and C is defined by the matrix

$$
\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
$$

Since C is orthogonal (it has determinant -1), F is an isometry.

A 'first principles' computation also shows directly that F is an isometry.

(d) If we compare $\|F(\mathbf{p}) - F(\mathbf{q})\|$ with $\|\mathbf{p} - \mathbf{q}\|$, we get

$$
\begin{aligned}
\|F(\mathbf{p}) - F(\mathbf{q})\| &= \|(p_1, p_2, 1) - (q_1, q_2, 1)\| \\
&= \|(p_1 - q_1, p_2 - q_2, 0)\| \\
&= \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} \\
\|\mathbf{p} - \mathbf{q}\| &= \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2}.
\end{aligned}
$$

We can now see that, if $p_3 - q_3 \neq 0$, these two results are different.

Hence F is not an isometry.

There is a directly geometric way of seeing that F cannot be an isometry. It maps the whole of $\mathbf{E}^3$ onto the *plane*

$$
z = 1.
$$

Therefore, some points which were separate in the domain have the same images in the codomain. Isometries have to be one–one, so F cannot be an isometry.

Solution 2.1

The best thing to do is probably to cast the translation into the standard form for isometries and apply the results of the section.

Informally, a translation has no orthogonal part. Formally, this means that a translation is an isometry for which

$$
C = I,
$$

the identity matrix. It follows that T_* is represented by the identity matrix.

It follows that the vector part of $T_*(\mathbf{v_p})$ is

$$
I(\mathbf{v}) = \mathbf{v}.
$$

The result follows.

Solution 2.2

If we use the notation of the remark after Theorem 2.3, we have

$$A=\begin{pmatrix}\frac{2}{3} & \frac{2}{3} & \frac{1}{3}\\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3}\\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3}\end{pmatrix},$$

$$B=\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\end{pmatrix},$$

$${}^tB=\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\end{pmatrix}.$$

$$\begin{aligned}C &= {}^tB\,A\\ &=\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{2}{3} & \frac{2}{3} & \frac{1}{3}\\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3}\\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3}\end{pmatrix}\\ &=\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3}\\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}}\end{pmatrix}.\end{aligned}$$

Since

$$C(\mathbf{p})=\begin{pmatrix}0\\ \frac{1}{3}\\ \frac{4}{3\sqrt{2}}\end{pmatrix},$$

we require translation by

$$\begin{aligned}\mathbf{q}-C(\mathbf{p}) &= \left(3-0,-1-\frac{1}{3},1-\frac{4}{3\sqrt{2}}\right)\\ &= \left(3,-\frac{4}{3},1-\frac{4}{3\sqrt{2}}\right).\end{aligned}$$

Solution 3.1

The first frame has attitude matrix

$$\begin{pmatrix}\frac{2}{3} & \frac{2}{3} & \frac{1}{3}\\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3}\\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3}\end{pmatrix}$$

and so its orientation is obtained from the determinant whose value is

$$\begin{aligned}&\frac{2}{3}\left(\frac{1}{3}\frac{2}{3}+\frac{2}{3}\frac{2}{3}\right)-\frac{2}{3}\left(-\frac{2}{3}\frac{2}{3}-\frac{2}{3}\frac{1}{3}\right)+\frac{1}{3}\left(\frac{2}{3}\frac{2}{3}-\frac{1}{3}\frac{1}{3}\right)\\ &= \frac{1}{27}(2\times 6-2\times(-6)+1\times 3)\\ &= \frac{1}{27}\times 27\\ &= 1.\end{aligned}$$

Thus the frame is positively oriented.

For the second frame, the attitude matrix is

$$\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\end{pmatrix}$$

with determinant

$$\begin{aligned}&\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}-0\right)-0+\frac{1}{\sqrt{2}}\left(0-\frac{1}{\sqrt{2}}\right)\\ &= -\frac{1}{2}-\frac{1}{2}\\ &= -1.\end{aligned}$$

Hence the frame is negatively oriented.

The isometry found in the earlier exercise has the following matrix for its orthogonal part.

$$\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3}\\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}}\end{pmatrix}$$

The determinant of this matrix is

$$\begin{aligned}&\frac{1}{\sqrt{2}}\left(-\frac{1}{9\sqrt{2}}-\frac{8}{9\sqrt{2}}\right)-0+\frac{1}{\sqrt{2}}\left(-\frac{8}{9\sqrt{2}}-\frac{1}{9\sqrt{2}}\right)\\ &= -\frac{1}{2}-\frac{1}{2}\\ &= -1.\end{aligned}$$

This fits in with the calculations of the frames' orientations. Since the isometry carries a positively oriented frame to a negatively oriented one, its sign must be −1.

Solution 3.2

Firstly, we have

$$\mathbf{v}\times\mathbf{w}=(-2,0,-6).$$

Now we calculate the various images.

$$\begin{pmatrix}-\frac{2}{3} & \frac{2}{3} & -\frac{1}{3}\\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}3\\ 1\\ -1\end{pmatrix}=\begin{pmatrix}-1\\ 3\\ 1\end{pmatrix};$$

$$\begin{pmatrix}-\frac{2}{3} & \frac{2}{3} & -\frac{1}{3}\\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}-3\\ -3\\ 1\end{pmatrix}=\begin{pmatrix}-\frac{1}{3}\\ -\frac{11}{3}\\ -\frac{7}{3}\end{pmatrix};$$

$$\begin{pmatrix}-\frac{2}{3} & \frac{2}{3} & -\frac{1}{3}\\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}-2\\ 0\\ -6\end{pmatrix}=\begin{pmatrix}\frac{10}{3}\\ \frac{8}{3}\\ -\frac{14}{3}\end{pmatrix}.$$

Since the isometry defined by C has orthogonal part C, we have

$$C_* = C$$

and, so,

$$\operatorname{sgn} C_* = \det C.$$

Now, calculating the determinant of C shows that

$$\operatorname{sgn} C_* = -1.$$

Also, we have

$$\begin{aligned} C(\mathbf{v}) \times C(\mathbf{w}) &= \left(-\frac{10}{3}, -\frac{8}{3}, \frac{14}{3}\right) \\ &= -\left(\frac{10}{3}, \frac{8}{3}, -\frac{14}{3}\right) \\ &= -C(\mathbf{v} \times \mathbf{w}). \end{aligned}$$

Since $\operatorname{sgn} C_* = -1$, we have

$$\begin{aligned} C(\mathbf{v} \times \mathbf{w}) &= -C(\mathbf{v}) \times C(\mathbf{w}) \\ &= \operatorname{sgn} C_* C(\mathbf{v}) \times C(\mathbf{w}). \end{aligned}$$

This completes the check.

Solution 4.1

Our main weapon for tackling such a question is Theorem 4.2. If we set

$$\gamma = F(\beta),$$

then we have exactly the conditions specified in Theorem 4.2 and we may apply its conclusions.

(a) If β is a cylindrical helix, we know that

$$\frac{\tau_\beta}{\kappa_\beta} = a,$$

where a is a constant. But

$$\begin{aligned} \frac{\tau_\gamma}{\kappa_\gamma} &= \frac{\pm\tau_\beta}{\kappa_\beta} \\ &= \pm a, \end{aligned}$$

which is also a constant. Thus $\gamma = F(\beta)$ is also a cylindrical helix.

(b) Here we apply the definition of spherical image. The coordinate functions of $\tilde{\beta}$ are those of the unit tangent β'. Abusing notation slightly, we can write

$$\tilde{\beta} = \beta'.$$

Similarly,

$$\tilde{\gamma} = \gamma'.$$

Now we bring in the isometry F and its derivative map C.

$$\begin{aligned} \tilde{\gamma} &= \gamma' \\ &= (F(\beta))' \\ &= F_*(\beta') \\ &= C(\beta') \\ &= C(\tilde{\beta}). \end{aligned}$$

This is exactly what we were asked to show.

There is an alternative proof that you may prefer. We can write the spherical image of β as

$$\tilde{\beta} = T,$$

(with much the same sort of abuse of notation as above). Similarly, the spherical image of $F(\beta)$ is

$$\begin{aligned} \overline{T} &= F_*(T) \quad \text{(using Theorem 4.2)} \\ &= F_*(\tilde{\beta}) \\ &= C(\tilde{\beta}). \end{aligned}$$

Solution 4.2

In order to find $\overline{\alpha}$, we have to calculate

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \\ 2t \end{pmatrix} = \begin{pmatrix} -\cos t \\ \frac{1}{\sqrt{2}}(\sin t - 2t) \\ \frac{1}{\sqrt{2}}(\sin t + 2t) \end{pmatrix},$$

Thus

$$\overline{\alpha}(t) = \left(-\cos t, \frac{1}{\sqrt{2}}(\sin t - 2t), \frac{1}{\sqrt{2}}(\sin t + 2t)\right).$$

Since, with our usual mild abuse of notation, $C_* = C$, we can calculate $C_*(Y)$ as follows.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} t \\ 1 - t^2 \\ 1 + t^2 \end{pmatrix} = \begin{pmatrix} -t \\ -\sqrt{2}t^2 \\ \sqrt{2} \end{pmatrix},$$

so

$$\overline{Y} = C_*(Y) = (-t, -\sqrt{2}t^2, \sqrt{2}).$$

Now,

$$Y' = (1, -2t, 2t),$$
$$\overline{Y}' = (-1, -2\sqrt{2}t, 0)$$

and since

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ -2t \\ 2t \end{pmatrix} = \begin{pmatrix} -1 \\ -2\sqrt{2}t \\ 0 \end{pmatrix},$$

we have

$$\begin{aligned} C_*(Y') &= (-1, -2\sqrt{2}t, 0) \\ &= \overline{Y}'. \end{aligned}$$

We have

$$\begin{aligned} \alpha'(t) &= (-\sin t, \cos t, 2), \\ \alpha''(t) &= (-\cos t, -\sin t, 0), \\ \overline{\alpha}'(t) &= \left(\sin t, \frac{1}{\sqrt{2}}(\cos t - 2), \frac{1}{\sqrt{2}}(\cos t + 2)\right), \\ \overline{\alpha}''(t) &= \left(\cos t, -\frac{1}{\sqrt{2}}\sin t, -\frac{1}{\sqrt{2}}\sin t\right). \end{aligned}$$

Now

$$\begin{aligned} C(\alpha'') &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos t \\ -\frac{1}{\sqrt{2}}\sin t \\ -\frac{1}{\sqrt{2}}\sin t \end{pmatrix} \end{aligned}$$

so

$$C_*(\alpha'') = \overline{\alpha}''.$$

Finally,

$$\begin{aligned} Y' \cdot \alpha'' &= (1, -2t, 2t) \cdot (-\cos t, -\sin t, 0) \\ &= -\cos t + 2t \sin t. \\ \overline{Y}' \cdot \overline{\alpha}'' &= (-1, -2\sqrt{2}t, 0) \cdot \left(\cos t, -\frac{1}{\sqrt{2}} \sin t, -\frac{1}{\sqrt{2}} \sin t\right) \\ &= -\cos t + 2t \sin t \\ &= Y' \cdot \alpha''. \end{aligned}$$

Solution 5.1

In order to invoke our main theorem, we must find the curvature and torsion functions of both curves. They are actually unit-speed curves, but there is no harm in using the more general methods usually used for non unit-speed curves. We choose to use the unit-speed methods.

We begin with α.

$$\alpha'(t) = (-\sin t, \cos t, 0).$$

From this we can see that α is unit speed, since

$$\|\alpha'(t)\|^2 = \sin^2 t + \cos^2 t = 1.$$

Hence

$$\begin{aligned} T_\alpha &= (-\sin t, \cos t, 0); \\ T'_\alpha &= (-\cos t, -\sin t, 0), \\ \kappa_\alpha &= \|T'_\alpha\| \\ &= 1; \\ N_\alpha &= T'_\alpha / \kappa_\alpha \\ &= (-\cos t, -\sin t, 0); \\ B_\alpha &= T_\alpha \times N_\alpha \\ &= (0, 0, 1); \\ B'_\alpha &= (0, 0, 0), \\ \tau_\alpha &= -B'_\alpha \cdot N_\alpha \\ &= 0. \end{aligned}$$

Next, we deal with β.

$$\beta'(t) = \frac{1}{3}(2 \sin t + 2 \cos t, -2 \sin t + \cos t, -\sin t + 2 \cos t).$$

$$\begin{aligned} \|\beta'(t)\|^2 &= \frac{1}{9}(4 \sin^2 t + 4 \cos^2 t + 4 \sin^2 t + \cos^2 t \\ &\quad + \sin^2 t + 4 \cos^2 t) \\ &= 1. \end{aligned}$$

Thus β is also unit speed.

$$\begin{aligned} T_\beta &= \frac{1}{3}(2 \sin t + 2 \cos t, -2 \sin t + \cos t, -\sin t + 2 \cos t); \\ T'_\beta &= \frac{1}{3}(2 \cos t - 2 \sin t, -2 \cos t - \sin t, -\cos t - 2 \sin t), \\ \kappa_\beta &= \|T'_\beta\| \\ &= \frac{1}{3}\sqrt{9 \cos^2 t + 9 \sin^2 t} \\ &= 1; \\ N_\beta &= T'_\beta / \kappa_\beta \\ &= \frac{1}{3}(2 \cos t - 2 \sin t, -2 \cos t - \sin t, -\cos t - 2 \sin t); \\ B_\beta &= T_\beta \times N_\beta \\ &= \frac{1}{9}(3, 6, -6) \\ &= \frac{1}{3}(1, 2, -2); \\ B'_\beta &= 0, \\ \tau_\beta &= -B'_\beta \cdot N_\beta \\ &= 0. \end{aligned}$$

From the above we can see that the two curves satisfy the requirements of the main theorem and are, therefore, congruent.

To find the isometry, we calculate the Frenet frame at $t = 0$ for both curves.

For α:

$$\begin{aligned} T_\alpha(0) &= (0, 1, 0), \\ N_\alpha(0) &= (-1, 0, 0), \\ B_\alpha(0) &= (0, 0, 1). \end{aligned}$$

For β:

$$\begin{aligned} T_\beta(0) &= \frac{1}{3}(2, 1, 2), \\ N_\beta(0) &= \frac{1}{3}(2, -2, -1), \\ B_\beta(0) &= \frac{1}{3}(1, 2, -2). \end{aligned}$$

Since the torsions of the two curves are *equal*, we require an isometry that will carry the Frenet frame at $\alpha(0)$ to the Frenet frame at $\beta(0)$, and also map $\alpha(0)$ to $\beta(0)$. The attitude matrices of the Frenet frames are

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix}$$

respectively.

If we apply the ‘${}^tB\,A$’ formula from Section 2, then the orthogonal part of the isometry is

$$\begin{aligned} C &= \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix}. \end{aligned}$$

We also have

$$\begin{aligned} \alpha(0) &= (1, 0, 0), \\ \beta(0) &= \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right). \end{aligned}$$

Next we calculate $C(\alpha(0))$.

$$\begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}.$$

$$C(\alpha(0) = \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right).$$

The translation part of the isometry must be by the vector

$$\begin{aligned}\beta(0) - C(\alpha(0)) &= \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right) - \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)\\ &= (1, -1, 0).\end{aligned}$$

Since the orthogonal and translation parts have been found, the isometry is completely specified. We can give a description as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Solution 5.2

We use the hint offered and give only the calculations for β. Since the hint implies that neither is unit speed, we use the general formulas.

First we calculate speed, curvature and torsion.

$$\begin{aligned}\beta'(t) &= (3\sqrt{2}, -6t, 3\sqrt{2}t^2),\\ \beta''(t) &= (0, -6, 6\sqrt{2}t),\\ \beta'''(t) &= (0, 0, 6\sqrt{2}).\end{aligned}$$

$$\begin{aligned}\beta'(t) \times \beta''(t) &= (-18\sqrt{2}t^2, -36t, -18\sqrt{2})\\ &= -18\sqrt{2}(t^2, \sqrt{2}t, 1).\end{aligned}$$

$$\beta'(t) \times \beta''(t) \cdot \beta'''(t) = -216.$$

$$\begin{aligned}v_\beta &= \|\beta'(t)\|\\ &= \sqrt{18 + 36t^2 + 18t^4}\\ &= \sqrt{18}\sqrt{1 + 2t^2 + t^4}\\ &= 3\sqrt{2}(1 + t^2).\end{aligned}$$

$$\begin{aligned}\|\beta'(t) \times \beta''(t)\| &= 18\sqrt{2}\sqrt{1 + 2t^2 + t^4}\\ &= 18\sqrt{2}(1 + t^2).\end{aligned}$$

$$\begin{aligned}\kappa_\beta &= \frac{\|\beta'(t) \times \beta''(t)\|}{v_\beta^3}\\ &= \frac{18\sqrt{2}(1 + t^2)}{54\sqrt{2}(1 + t^2)^3}\\ &= \frac{1}{3(1 + t^2)^2}.\end{aligned}$$

$$\begin{aligned}\tau_\beta &= \frac{\beta'(t) \times \beta''(t) \cdot \beta'''(t)}{\|\beta'(t) \times \beta''(t)\|^2}\\ &= \frac{-216}{18^2 \times 2 \times (1 + t^2)^2}\\ &= -\frac{1}{3(1 + t^2)^2}.\end{aligned}$$

Comparing these with the results found by *O'Neill*, we see that

$$v_\alpha = v_\beta > 0,\ \kappa_\alpha = \kappa_\beta,\ \tau_\alpha = -\tau_\beta.$$

Thus the curves are congruent.

To find the isometry we require the Frenet frames at $t = 0$. There is little point in finding the general forms of the Frenet frame for β, we might as well just calculate the values of the expressions required by Chapter II, Theorem 4.3 at $t = 0$ and use them to find the frame at this point only.

For α we have the results from Chapter II available and they yield the following.

$$\begin{aligned}T_\alpha(0) &= \frac{1}{\sqrt{2}}(1, 0, 1),\\ N_\alpha(0) &= (0, 1, 0),\\ B_\alpha(0) &= \frac{1}{\sqrt{2}}(-1, 0, 1).\end{aligned}$$

For β we proceed as indicated above, using the results already obtained.

$$\begin{aligned}T_\beta(0) &= \frac{\beta'(0)}{v_\beta(0)}\\ &= \frac{1}{3\sqrt{2}}3\sqrt{2}(1, 0, 0)\\ &= (1, 0, 0);\\ B_\beta(0) &= \frac{\beta'(0) \times \beta''(0)}{\|\beta'(0) \times \beta''(0)\|}\\ &= \frac{1}{18\sqrt{2}}18(0, 0, -\sqrt{2})\\ &= (0, 0, -1);\\ N_\beta(0) &= B_\beta(0) \times T_\beta(0)\\ &= (0, -1, 0).\end{aligned}$$

Because $\tau_\alpha = -\tau_\beta$, we want

$$\left.\begin{matrix} T_\alpha(0) \\ N_\alpha(0) \\ B_\alpha(0) \end{matrix}\right\} \longmapsto \left\{\begin{matrix} T_\beta(0) \\ N_\beta(0) \\ -B_\beta(0) \end{matrix}\right. .$$

(Note the minus sign.)

The respective attitude matrices of these frames are

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

since $-B_\beta(0) = (0, 0, 1)$.

The orthogonal part C of the isometry is, therefore, given by

$$\begin{aligned}C &= {}^t\!B\,A\\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.\end{aligned}$$

It remains to find only the translation part. We have

$$\alpha(0) = \beta(0) = (0, 0, 0).$$

Since

$$C(\alpha(0)) = (0, 0, 0) = \beta(0),$$

the translation part is the identity and the isometry is just the orthogonal transformation defined by the matrix C.

Solution 5.3

At first sight the condition given in the question looks somewhat mysterious. Let us begin by looking at the first half of the following assertion.

If α and β are congruent, then β can be expressed as indicated.

We start by writing down what we know. Using the usual notation,

$$\alpha(t) = \alpha_1(t)(1,0,0) + \alpha_2(t)(0,1,0) + \alpha_3(t)(0,0,1).$$

We also know that there is an isometry $F = TC$ such that

$$\beta = F(\alpha).$$

If we set

$$\begin{aligned} \mathbf{e}_1 &= C(1,0,0), \\ \mathbf{e}_2 &= C(0,1,0), \\ \mathbf{e}_3 &= C(0,0,1), \end{aligned}$$

then, by the linearity of the orthogonal part,

$$C(\alpha) = \alpha_1(t)\mathbf{e}_1 + \alpha_2(t)\mathbf{e}_2 + \alpha_3(t)\mathbf{e}_3.$$

Since the $\mathbf{e}_i$s are images of an orthonormal basis under an orthogonal transformation, it follows that they are also orthonormal, that is

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}.$$

If we define $\mathbf{p}$ to be $T(0,0,0)$, then

$$\begin{aligned} \beta &= F(\alpha) \\ &= \mathbf{p} + C(\alpha) \\ &= \mathbf{p} + \alpha_1(t)\mathbf{e}_1 + \alpha_2(t)\mathbf{e}_2 + \alpha_3(t)\mathbf{e}_3. \end{aligned}$$

This completes the proof of the first half.

The first half of the proof also indicates how to prove the other half. Suppose now that β can be written as

$$\beta = \mathbf{p} + \alpha_1(t)\mathbf{e}_1 + \alpha_2(t)\mathbf{e}_2 + \alpha_3(t)\mathbf{e}_3.$$

Let T be translation by $\mathbf{p}$ and let C be the (unique) orthogonal transformation that maps

$$\left.\begin{matrix} (1,0,0) \\ (0,1,0) \\ (0,0,1) \end{matrix}\right\} \longmapsto \left\{\begin{matrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{matrix}\right. .$$

Then, from the way we have defined F and the linearity of C,

$$\beta = F(\alpha)$$

and the curves are congruent.

Solution 5.4

We begin by calculating the curvature and torsion of β.

$$\begin{aligned} \beta'(t) &= (1 + \sqrt{3}\cos t, -2\sin t, \sqrt{3} - \cos t), \\ \beta''(t) &= (-\sqrt{3}\sin t, -2\cos t, \sin t), \\ \beta'''(t) &= (-\sqrt{3}\cos t, 2\sin t, \cos t); \end{aligned}$$

$$\beta'(t) \times \beta''(t) = (2\sqrt{3}\cos t - 2, -4\sin t, -2\sqrt{3} - 2\cos t),$$

$$\beta'(t) \times \beta''(t) \cdot \beta'''(t) = -8;$$

$$\begin{aligned} v_\beta &= \|\beta'(t)\| \\ &= \sqrt{4+4} \\ &= 2\sqrt{2}; \end{aligned}$$

$$\begin{aligned} \kappa_\beta &= \frac{\|\beta'(t) \times \beta''(t)\|}{v_\beta^3} \\ &= \frac{4\sqrt{2}}{16\sqrt{2}} \\ &= \frac{1}{4}; \end{aligned}$$

$$\begin{aligned} \tau_\beta &= \frac{\beta'(t) \times \beta''(t) \cdot \beta'''(t)}{\|\beta'(t) \times \beta''(t)\|^2} \\ &= \frac{-8}{32} \\ &= -\frac{1}{4}. \end{aligned}$$

Since both the curvature and torsion are constant, so is their ratio and β is a cylindrical helix. Because the curvature is constant, it is a circular helix.

In order to find the values of a and b that will make α congruent to β, we calculate the speed, curvature and torsion for α.

$$\begin{aligned} \alpha'(t) &= (-a\sin t, a\cos t, b), \\ \alpha''(t) &= (-a\cos t, -a\sin t, 0), \\ \alpha'''(t) &= (a\sin t, -a\cos t, 0), \end{aligned}$$

$$\begin{aligned} \alpha'(t) \times \alpha''(t) &= (ab\sin t, -ab\cos t, a^2), \\ \alpha'(t) \times \alpha''(t) \cdot \alpha'''(t) &= a^2 b. \end{aligned}$$

$$\begin{aligned} \mathbf{v}_\alpha &= \|\alpha'(t)\| \\ &= \sqrt{a^2 + b^2}, \end{aligned}$$

$$\begin{aligned} \kappa_\alpha &= \frac{\|\alpha'(t) \times \alpha''(t)\|}{v_\alpha^3} \\ &= \frac{a\sqrt{a^2+b^2}}{(a^2+b^2)\sqrt{a^2+b^2}} \\ &= \frac{a}{a^2+b^2}; \end{aligned}$$

$$\begin{aligned} \tau_\alpha &= \frac{\alpha'(t) \times \alpha''(t) \cdot \alpha'''(t)}{\|\alpha'(t) \times \alpha''(t)\|^2} \\ &= \frac{a^2 b}{a^2(a^2+b^2)} \\ &= \frac{b}{a^2+b^2}. \end{aligned}$$

We require

$$\begin{aligned} v_\alpha &= v_\beta \\ \kappa_\alpha &= \kappa_\beta. \end{aligned}$$

The first requirement forces

$$a^2 + b^2 = 8$$

and so the second then gives

$$a = 2.$$

It follows that

$$b^2 = 4.$$

Either of the solutions for b would be acceptable, taking

$$b = -2$$

actually gives

$$\tau_\alpha = \tau_\beta,$$

which makes finding the isometry slightly easier. Note that this ambiguity is to be expected, depending on whether we require α to be a helix of the same 'handedness' as β or not.

With the choice of b as above, we can now find the Frenet frames at $t = 0$ and, hence, the isometry.

For β we have

$$\begin{aligned} \beta'(0) &= (1 + \sqrt{3}, 0, \sqrt{3} - 1), \\ T_\beta(0) &= \frac{1}{2\sqrt{2}}(1 + \sqrt{3}, 0, \sqrt{3} - 1), \end{aligned}$$

$$
\begin{aligned}
B_\beta(0) &= \frac{\beta'(0) \times \beta''(0)}{\|\beta'(0) \times \beta''(0)\|} \\
&= \frac{1}{2\sqrt{2}}(\sqrt{3}-1, 0, -\sqrt{3}-1), \\
N_\beta(0) &= B_\beta(0) \times T_\beta(0) \\
&= \frac{1}{8}(0, -8, 0) \\
&= (0, -1, 0).
\end{aligned}
$$

The corresponding results for α are

$$
\begin{aligned}
\alpha'(0) &= (0, -2, 0), \\
T_\alpha(0) &= \frac{1}{\sqrt{2}}(0, 1, -1), \\
B_\alpha(0) &= \frac{\alpha'(0) \times \alpha''(0)}{\|\alpha'(0) \times \alpha''(0)\|} \\
&= \frac{1}{\sqrt{2}}(0, 1, 1), \\
N_\alpha(0) &= B_\alpha(0) \times T_\alpha(0) \\
&= (-1, 0, 0).
\end{aligned}
$$

If we let $F = TC$ be the isometry such that

$$F(\alpha) = \beta,$$

then the attitude matrices of the relevant frames are

$$
A = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}
$$

and

$$
B = \begin{pmatrix} \frac{1+\sqrt{3}}{2\sqrt{2}} & 0 & \frac{\sqrt{3}-1}{2\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{\sqrt{3}-1}{2\sqrt{2}} & 0 & \frac{-\sqrt{3}-1}{2\sqrt{2}} \end{pmatrix}.
$$

Note that there is no sign reversal because the torsions are equal.

The orthogonal part of the isometry is given by $C = {}^t\!B\,A$, noting that, in this case, ${}^t\!B = B$.

$$
\begin{aligned}
C &= {}^t\!B\,A \\
&= \begin{pmatrix} \frac{1+\sqrt{3}}{2\sqrt{2}} & 0 & \frac{\sqrt{3}-1}{2\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{\sqrt{3}-1}{2\sqrt{2}} & 0 & \frac{-\sqrt{3}-1}{2\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\alpha(0) &= (2, 0, 0), \\
\beta(0) &= (0, 2, 0).
\end{aligned}
$$

Since

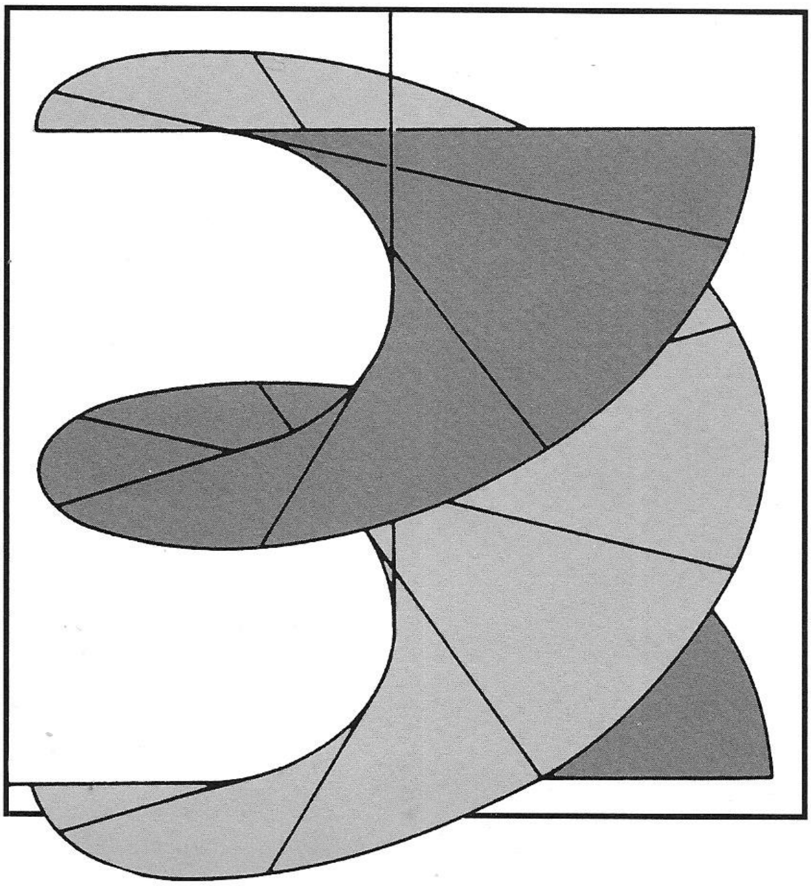
$$C(\alpha(0)) = (0, 2, 0) = \beta(0)$$

the translation part is the identity. Thus F is the orthogonal transformation defined by C.

ISBN 0 7492 4772 X

M434 Part IV
Mathematics: A Fourth Level Course

M434
DIFFERENTIAL GEOMETRY



PART IV
CALCULUS ON A SURFACE

The Open University

The Open University

Mathematics: A Fourth Level Course

M434 Differential Geometry

Part IV Calculus on a Surface

Prepared for the Course Team
by Bob Margolis

Set book

Barrett O'Neill, *Elementary Differential Geometry*, hardback edition (Academic Press, 1966).

It is essential to have this book; the course is based on it and will not make sense without it.

The set book is referred to as *O'Neill*.

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published as *M334* 1976.

First published as *M434* 1993. Reprinted 1996.

Copyright © 1993 The Open University

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, without written permission from the publisher or a licence from the Copyright Licensing Agency Limited. Details of such licences (for reprographic reproduction) may be obtained from the Copyright Licensing Agency Ltd of 90 Tottenham Court Road, London, W1P 9HE.

Edited, designed and typeset by the Open University using the Open University TEX System.

Printed in the United Kingdom by Hobbs the Printers Ltd, Totton, Hampshire.

ISBN 0 7492 4773 8

This text forms part of an Open University Fourth Level Course. If you would like a copy of *Studying with The Open University*, please write to the Central Enquiry Service, PO Box 200, The Open University, Walton Hall, Milton Keynes, MK7 6YZ. If you have not already enrolled on the Course and would like to buy this or other Open University material, please write to Open University Educational Enterprises Ltd, 12 Cofferidge Close, Stony Stratford, Milton Keynes, MK11 1BY, United Kingdom.

1.2

Contents

Introduction 4

1 Surfaces 5

2 Patch computations 8

3 Functions and tangent vectors 11

4 Differential forms on a surface 17

5 Mappings of surfaces 22

6 Topological properties 26

7 Summary 28

8 Appendix: integration 29

Solutions to the exercises 32

Introduction

We now begin the study of surfaces. We shall tackle surfaces using essentially the same approach as we did with curves. We equip the surface being studied with a frame field and then use the rate of change of the frame field to give geometric information about the shape of the surface.

On curves, we can move only along the curve, so derivatives of the Frenet frame were 'ordinary' derivatives with respect to the parameter (t or s) used to define the curve. On a surface there are two 'degrees of freedom'; at any point there are two independent directions in which we could move. This corresponds to the intuitive idea that surfaces have tangent planes (two-dimensional) rather than tangent lines (one-dimensional) as curves do.

We have already defined most of the tools we shall need. The covariant derivatives connection forms and structural equations give us the methods for discussing rates of change when we can move in arbitrary directions. We need to consider what happens to these general ideas when we restrict ourselves to moving about on a surface.

Part II, Sections 5–8.

Note: The section numbering below is that of the *text*. There are no text sections corresponding to IV.6 and IV.8 of *O'Neill*.

We begin, in Section 1, by defining what we mean by a surface in $\mathbf{E}^3$. Unlike curves, surfaces will be subsets of $\mathbf{E}^3$. We shall require surfaces to be 'two-dimensional' and 'smooth'. The smoothness will be ensured (as it was for curves) by requiring any functions involved in the definition to be differentiable.

Curves, you will recall, were functions.

As usual, 'differentiable' means differentiable infinitely many times.

To formalize the notion of a surface being two-dimensional, we ask that near each point the surface 'looks like' a plane. By 'looks like' we mean that surrounding each point there is a region which is the image of a part of the plane $\mathbf{E}^2$ under a well-behaved (differentiable) mapping.

Unlike curves, we do not ask for a single function to produce the whole of a surface. If you would like an analogy, surfaces are defined in much the same way as children make papier mâché masks: built up of small, overlapping pieces, each of which is a distorted piece of plane. The distortion must be gentle enough to be done by a differentiable function and the pieces must overlap smoothly with no sharp points or ridges.

The *definition* of surface turns out to be a little unwieldy for practical use. We would expect that the set obtained by drawing the graph of a differentiable function

$$f : \mathbf{E}^2 \longrightarrow \mathbf{R},$$

that is, the set

$$M = \{(p_1, p_2, p_3) \in \mathbf{E}^3 : p_3 = f(p_1, p_2)\},$$

to be a surface. We would also expect that the set of points satisfying a relation such as

$$x^2 + y^2 + z^2 = 1$$

(which defines a sphere) to be a surface. The next step after the definition is to prove a theorem that gives a practical test to determine whether a given subset of $\mathbf{E}^3$ is a surface.

The section concludes with some examples.

Section 2 shows how a system of coordinates can be transferred from the plane $\mathbf{E}^2$ to a surface and provides some further examples of surfaces. The system of coordinates will be the starting point for defining a frame field adapted to the surface.

Section 3 is fairly technical in nature. We take some of the concepts of *Part I*: real-valued functions, tangent vectors and vector fields and show how they may be defined on a surface. We also show how the concept of differentiability can be defined for functions whose domain is a surface rather than the real numbers.

Section 4 continues the work of Section 3 by defining what are meant by differential forms and exterior derivative on a surface. We need these definitions because they are essential for the connection forms and Cartan structural equations. As these equations are the analogues of the Frenet formulas, we are expecting to extract geometric information about surfaces from them.

In Section 5, we consider mappings between surfaces. We do so for two reasons. One is, simply, that the possibility of mappings between surfaces exists, so curiosity demands that we investigate! The other is that mappings from a 'standard' surface to a new one can give geometric information about the new surface.

Our 'standard' surface will be a sphere.

Section 6 discusses some properties that surfaces may possess. The three considered: connectedness, compactness and orientability are grouped together because they are topological properties and do not depend directly on notions of differentiability. However, the only one we shall really consider is orientability.

Finally, Section 7 provides a summary.

Note: Sections IV.6 and IV.8 of *O'Neill* do not form part of *M434*. However, Section 6 does provide a link with the integral calculus that you have already studied in other courses. For this reason, there is an appendix to this text which discusses Section IV.6. **The appendix is optional and is not assessed in any way.**

Section IV.8 provides a generalization of the concept of a surface which is important for further work but which is not needed for *M434*.

Study advice

The following represents a possible plan for study weeks.

Week 1 *O'Neill*, Chapter IV, Sections 1–3.

Week 2 *O'Neill*, Chapter IV, Sections 4, 5 and 6.

This leaves 2 study weeks for the work on the first three sections of Chapter V and TMA03. You may well find tackling Sections 4, 5 and 6 in one week a rather heavier than average workload.

1 Surfaces

Read *O'Neill: Chapter IV, Section 1, pages 124–131.*

In this section we begin the programme outlined in the introduction by defining a surface in $\mathbf{E}^3$.

Definition 1.1 The name *coordinate patch* should remind you of the papier mâché analogy used in the introduction. This definition is very much a two-dimensional analogue of the definition of curve.

In intuitive terms, the one–one requirement ensures that the image $\mathbf{x}(D)$ of D cannot cross itself. Since we require $\mathbf{x}$ to be a mapping, it must be differentiable. This ensures that $\mathbf{x}(D)$ is a smoothly distorted piece of plane with no sharp peaks or ridges. We comment on the purpose of regularity below.

Proper patches It is not entirely clear from *O'Neill* why this idea is introduced. The reason is that requiring $\mathbf{x}$ to be a proper patch is enough to ensure that the image set $\mathbf{x}(D)$ is an open set. You may be aware of this from other courses, if not it does not matter.

We want $\mathbf{x}(D)$ to be an open set for Definition 1.2 to work properly.

There is a hint, in Fig. 4.1, as to why 'coordinate' appears in the definition. As usual, we assume that $\mathbf{E}^2$ is provided with Cartesian coordinates. These provide a 'coordinate grid' on the subset D which become a grid of curves on the image set $\mathbf{x}(D)$.

The requirement for regularity of $\mathbf{x}$ ensures that the grid on the image will be a *grid* and not just one set of curves. At each point on D we can place two independent tangent vectors, one along each grid line. Since the images of these tangent vectors must be independent (by regularity), these images will be tangent to the 'grid curves' in the image.

Thus $\mathbf{x}$ provides a 'curvilinear' coordinate system on the image set $\mathbf{x}(D)$, hence 'coordinate patch'.

Definition 1.2 This is the definition that formalizes the papier mâché construction process for surfaces, although it is actually phrased as a test to apply to a set to detect if the set is a surface.

You may find it useful to consider Definition 1.2 from the point of view of a creature living on the surface. The definition asserts that, wherever you are on the surface, you are within a set that looks like a smoothly distorted piece of plane. Because there is a neighbourhood surrounding you that is in the image of a proper patch, you cannot be on the edge of a distorted piece of plane.

Note that a given surface may consist of the image sets of several different proper patches, as the examples in *O'Neill* show. However, where patches overlap they cannot just meet at a point or line. To see how patches overlap, suppose that the point $\mathbf{p}$ of a surface M is in the image sets of both $\mathbf{x}_1$ and $\mathbf{x}_2$. Then there is a neighbourhood N_1 of $\mathbf{p}$ contained in the image of $\mathbf{x}_1$ and a neighbourhood N_2 of $\mathbf{p}$ contained in the image of $\mathbf{x}_2$. The intersection of two neighbourhoods is a neighbourhood. (Because of the way *O'Neill* defines 'neighbourhood' in M the intersection of two neighbourhoods will be the smaller one.)

That is, where the image sets of patches overlap.

Points and lines cannot contain neighbourhoods of any of their points, so patches must overlap in a whole region. This will be important for some of the later, technical theorems.

The sphere Notice how the definition of surface in terms of *proper* patches makes for difficulties in verifying that the sphere is, indeed, a surface.

The first patch defined in *O'Neill* cannot include the equator without breaking the condition that the image is an open set in the surface.

Note that the matrix computed *is* the transpose of the Jacobian as defined earlier in the course. The domain of the patch $\mathbf{x}$ is a subset of $\mathbf{E}^2$ and has dimension 2. The rank of the Jacobian is the same as the dimension of the domain and so $\mathbf{x}$ is regular.

Notation This example introduces some notation that will be used consistently for surfaces. When we are dealing with $\mathbf{E}^2$, or an open subset D of $\mathbf{E}^2$, as the domain of a patch, we shall refer to the coordinate functions as u and v rather than x and y. A consequence is that partial derivatives appear as

$$\frac{\partial}{\partial u} \quad \text{and} \quad \frac{\partial}{\partial v}$$

and the 1-forms appear as

$$du \quad \text{and} \quad dv.$$

This does avoid confusion with the use of $\mathbf{x}$ for patches. However, it raises the possibility of confusion with $\mathbf{u}$ and $\mathbf{v}$ for tangent vectors. You may need to be quite careful in your work to indicate tangent vectors by underlining (where the printed text uses bold type).

Theorem 1.4 The difficulty in checking something as straightforward as a sphere makes something like this theorem highly desirable. You are not expected to follow *O'Neill*'s outline proof because we do not expect you to have met the Implicit Function Theorem. You are expected to be able to apply the test provided by the theorem.

Note that $dg = 0$ at a point $\mathbf{p}$ if, and only if,

$$\frac{\partial g}{\partial x}(\mathbf{p}) = \frac{\partial g}{\partial y}(\mathbf{p}) = \frac{\partial g}{\partial z}(\mathbf{p}) = 0.$$

Surfaces of revolution Note that the definition given by *O'Neill* is very restrictive: the profile curve may not cross (or meet) the axis of revolution. That means that, contrary to intuition, the sphere

$$x^2 + y^2 + z^2 = 1$$

is not a surface of revolution!

The definition is made restrictive to avoid the possibility of the 'surface' having a sharp point (if the profile curve meets the axis) or crossing itself (if the curve crosses the axis). The only 'safe' extension to the definition is where the profile curve is symmetric about the axis of revolution and crosses the axis at right angle (avoiding sharp points). This extension is dealt with in Exercise 12, page 133 of *O'Neill*.

Exercise 1.1 *O'Neill*, page 131, Exercise 1. This exercise is intended to reinforce your intuitive understanding of the definition of surface. You should try to describe where and why the definition breaks down in each case.

Exercise 1.2 *O'Neill*, page 132, Exercise 4.

Exercise 1.3 *O'Neill*, page 132, Exercise 5.

Exercise 1.4 *O'Neill*, page 132, Exercise 8.

Exercise 1.5 This exercise concerns the subset M of $\mathbf{E}^3$ defined by

$$M : z = \frac{x^2 - y^2}{4}$$

and the mapping $\mathbf{x} : \mathbf{E}^2 \longrightarrow \mathbf{E}^3$ defined by

$$\mathbf{x}(u, v) = (u + v, u - v, uv).$$

(a) By expressing M in the form $M : g = c$, or otherwise, show that M is a surface in $\mathbf{E}^3$.

(b) Show that $\mathbf{x}(u, v)$ belongs to M for all values of u and v.

(c) Show that $\mathbf{x}$ is onto M by showing that every point in M can be expressed in the form $\mathbf{x}(u, v)$ for suitable values of u and v.

(d) Show that $\mathbf{x}$ is a proper patch.

[*Solutions on page 32*]

2 Patch computations

Read *O'Neill: Chapter IV, Section 2, pages 133–140.*

Erratum *O'Neill*, page 136, the calculation of $r^{-2}g(\mathbf{x})$ should read:

$$\begin{aligned} r^{-2}g(\mathbf{x}) &= (\cos v \cos u)^2 + (\cos v \sin u)^2 + \sin^2 v \\ &= \cos^2 v + \sin^2 v = 1. \end{aligned} \qquad \blacksquare$$

In this section we show how patches provide a system of coordinates on a surface. We also start the process of abandoning patches as being rather restricting to work with and transfer our attention to parametrizations. These, roughly speaking, are patches with the one–one condition dropped.

Parameter curves The idea behind the notion of parameter curves is simple: they are the images of the coordinate grid in the domain of a patch.

There is a tight link between coordinates in $\mathbf{E}^2$, or some open subset D of $\mathbf{E}^2$, and the parameter curves on a surface. Suppose that

$$\mathbf{x} : D \longrightarrow \mathbf{E}^3$$

is one of the patches defining a surface M. We can write the equation of a line through

$$(u_0, v_0)$$

in D, parallel to the x-axis as

$$\alpha(t) = (u_0, v_0) + t(1, 0).$$

The image of α under $\mathbf{x}$ is

$$\mathbf{x}(\alpha(t)).$$

By Definition 2.1, the velocity of $\mathbf{x}(\alpha(t))$ at $t = 0$ is

$$\mathbf{x}_u(u_0, v_0).$$

But, because $\mathbf{x}$ is a mapping, the velocity of $\mathbf{x}(\alpha(t))$ is the image of the velocity of α under the derivative map $\mathbf{x}_*$.

However, the velocity of α is

$$\alpha'(t) = (1, 0)$$

everywhere on α. Hence

$$\mathbf{x}_u(u_0, v_0) = \mathbf{x}_*(1, 0).$$

Using the notation of *O'Neill* for the coordinate functions of $\mathbf{x}$, the Jacobian matrix for $\mathbf{x}_*$ is

$$\begin{pmatrix} \dfrac{\partial x_1}{\partial u}(u_0, v_0) & \dfrac{\partial x_1}{\partial v}(u_0, v_0) \\ \dfrac{\partial x_2}{\partial u}(u_0, v_0) & \dfrac{\partial x_2}{\partial v}(u_0, v_0) \\ \dfrac{\partial x_3}{\partial u}(u_0, v_0) & \dfrac{\partial x_3}{\partial v}(u_0, v_0) \end{pmatrix}.$$

Matrix multiplication by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then provides the expressions for the partial velocities near the bottom of page 134.

Regularity The partial velocities provide a simple test for regularity of a mapping

$$\mathbf{x} : D \longrightarrow \mathbf{E}^3.$$

The partial velocities form the columns of the Jacobian for $\mathbf{x}_*$. Regularity requires these columns to be linearly independent which is the same as requiring the tangent vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ to be linearly independent.

On the other hand, the cross product of two non-zero vectors is non-zero precisely when the vectors are linearly independent. Thus, $\mathbf{x}$ is regular whenever

$$\mathbf{x}_u \times \mathbf{x}_v \neq 0$$

everywhere on the domain of $\mathbf{x}$.

On page 136, *O'Neill* expresses $\mathbf{x}_u \times \mathbf{x}_v$ as a (formal) determinant. In this expression, the U_i must be interpreted as being composites

$$U_i \circ \mathbf{x} : (u, v) \longmapsto U_i(\mathbf{x}(u, v))$$

so that all the functions appearing in the determinant have the same domain. (If this is not done, the U_i have domain $\mathbf{E}^3$, all other functions have domain part of $\mathbf{E}^2$.)

This is the first example of using the same notation for a function defined on a surface and the composite of such a function with a patch. This abuse of notation will recur from time to time.

Examples The examples of the cylinder, surfaces of revolution and the torus will reappear from time to time in our work on surfaces.

One simple version of the general cylinder is the circular cylinder where the curve C is a circle. For example, we could take

$$C : x^2 + y^2 = r^2$$

with parametrization

$$\alpha(t) = (r \cos t, r \sin t, 0), \quad t \in \mathbf{R}.$$

This gives a corresponding parametrization

$$\mathbf{x}(u, v) = (r \cos u, r \sin u, v)$$

of M.

The partial velocities are

$$\mathbf{x}_u(u, v) = (-r \sin u, r \cos u, 0),$$
$$\mathbf{x}_v(u, v) = (0, 0, 1).$$

Calculating the cross product,

$$\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) = (r \cos u, r \sin u, 0).$$

Because $\cos u$ and $\sin u$ cannot simultaneously be zero, the cross product is non-zero and the mapping $\mathbf{x}$ is regular, as predicted by the general argument in *O'Neill*.

You might like to note that the *tangent vector* $\mathbf{x}_u \times \mathbf{x}_v$ has the same coordinate functions as α. Therefore it points along the normal to the surface.

Actually, we ought to expect $\mathbf{x}_u \times \mathbf{x}_v$ always to be normal to the surface at $\mathbf{x}(u, v)$. This is because $\mathbf{x}_u$ and $\mathbf{x}_v$ are independent tangent vectors which are tangents (in the usual sense) to the surface and hence define the tangent plane to the surface at $\mathbf{x}(u, v)$. The cross product will be perpendicular to the tangent plane and so normal to the surface. We shall exploit this method of obtaining a normal to a surface later on.

Ruled surfaces This concept is discussed in *O'Neill* only in the exercises for this section. We think it is important enough to mention here.

The basic idea is a generalization of the way cylinders were described by moving a line along a plane curve, keeping the line at right angles to the plane of the curve. The generalization involves two relaxations, allowing general curves in $\mathbf{E}^3$ instead of plane curves and allowing more general rules than 'keep at right angles to the plane of the curve'.

An example of the method of construction may clarify the general procedure.

Consider the curve

$$\alpha(t) = (t, t^2, t^3), \quad t \in \mathbf{R}.$$

Suppose that we draw a line through each point of α in the direction of the tangent to α at that point, in other words, in the direction of α'. We have

$$\alpha'(t) = (1, 2t, 3t^2).$$

To get to a typical point on one of these lines we must go to a point on the curve and then along some multiple of the velocity vector. Thus we should define a mapping $\mathbf{x}$ by

$$\mathbf{x}(u, v) = \alpha(u) + v\alpha'(u).$$

As a check:

$$\mathbf{x}(u, 0) = \alpha(u)$$

and so the curve α lies entirely in the image of $\mathbf{x}$. Also

$$\mathbf{x}_v(u, v) = \alpha'(u)$$

and so the velocities of the v-parameter curves are constant and match the velocity of the curve α.

It is fairly clear from the form of the v-parameter curves that they are straight lines. However, we can check that they are straight lines in the very restricted sense that *O'Neill* defined in Chapter II. The acceleration can be obtained by differentiating $\mathbf{x}_v$ with respect to v:

$$\frac{\partial}{\partial v}\mathbf{x}_v(u, v) = 0.$$

Since the acceleration is zero, the v-parameter curves are, indeed, straight lines.

In this example we can give an explicit form for $\mathbf{x}$:

$$\begin{aligned}\mathbf{x}(u, v) &= (u, u^2, u^3) + v(1, 2u, 3u^2) \\ &= (u + v, u(u + 2v), u^2(u + 3v)).\end{aligned}$$

The partial velocities are given by

$$\begin{aligned}\mathbf{x}_u(u, v) &= (1, 2(u + v), 3u(u + 2v)), \\ \mathbf{x}_v(u, v) &= (1, 2u, 3u^2).\end{aligned}$$

The cross product of the partial velocities is

$$\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) = (-6u^2v, 6uv, -2v).$$

This product is only zero if

$$u = v = 0.$$

So, if we define D to be $\mathbf{E}^2$ with the origin deleted, then

$$\mathbf{x} : D \longrightarrow \mathbf{E}^3$$

defines a regular mapping.

If we restrict the domain further, to $u, v > 0$, then it is possible to show that $\mathbf{x}$ becomes a proper patch and hence that the image set is a surface.

The general form of the above construction requires two functions: one to take the place of α above and the other to define the direction of each line, taking the place of α' above.

The general parametrization of a ruled surface is of the form

$$\mathbf{x}(u, v) = \beta(u) + v\delta(u).$$

As another example, the surface M defined by

$$M : z = xy$$

This particular surface is called a *saddle surface.*

has a parametrization that shows that it is a ruled surface. If we take the Monge patch suggested by the definition of M

$$\mathbf{x}(u,v) = (u, v, uv),$$

then we can rewrite it as

$$\begin{aligned}\mathbf{x}(u,v) &= (u,0,0) + v(0,1,u)\\ &= \beta(u) + v\delta(u),\end{aligned}$$

where

$$\beta(u) = (u,0,0), \quad \text{and} \quad \delta(u) = (0,1,u).$$

The fact that $\mathbf{x}$ can also be written as

$$\mathbf{x}(u,v) = (0,v,0) + u(1,0,v)$$

suggests that both the u and v-parameter curves are straight lines. Thus M is an example of a 'doubly ruled' surface.

We shall come across a number of surfaces which are, in an intuitive sense, curved and yet, like M in the above example, are generated by straight lines.

Exercise 2.1 *O'Neill*, page 140, Exercise 1.

Exercise 2.2 *O'Neill*, page 140, Exercise 2.

Exercise 2.3 *O'Neill*, page 140, Exercise 5.

Exercise 2.4 *O'Neill*, page 141, Exercise 6. Ignore the last part.

Exercise 2.5 *O'Neill*, page 141, Exercise 7.

Exercise 2.6 *O'Neill*, page 142, Exercise 10. Ignore the request for a sketch.

Exercise 2.7 *O'Neill*, page 142, Exercise 11. Ignore the request for a sketch.

Exercise 2.8 *O'Neill*, page 143, Exercise 12. Last part as for the previous exercise.

[*Solutions on page 33*]

3 Functions and tangent vectors

Read *O'Neill: Chapter IV, Section 3, pages 143–149.*

Errata

1 *O'Neill*, page 148, the statement of Lemma 3.8 should read as follows.

If $M : g = c$ is a surface in $\mathbf{E}^3$ and if the gradient vector field

$$\nabla g = \sum \left(\frac{\partial f}{\partial x_i}\right) U_i$$

(considered only at points of M) is never zero, then it is a non-vanishing normal vector field on the entire surface M.

2 *O'Neill*, page 148, the first three lines of the proof of Lemma 3.8 should be deleted. ■

In this section we discuss the differentiability of functions whose domain or codomain is a surface (rather than all of $\mathbf{E}^2$ or $\mathbf{E}^3$) and the notion of a tangent vector being tangent to a surface.

Differentiability When we discussed the calculus of $\mathbf{E}^2$ and $\mathbf{E}^3$ (*Parts I* and *II*) one of the main requirements was that all functions should be differentiable. All definitions of differentiability were, ultimately, based on the differentiability of 'ordinary' functions from $\mathbf{R}$ to $\mathbf{R}$. We shall take a similar approach here. There are two cases to consider: one where the surface is the domain and the other where it is the codomain.

Looking at a particular version of the first case, suppose that we have a surface M and a function

$$f : M \longrightarrow \mathbf{R}.$$

Now M must be defined by one or more (proper) patches. If $\mathbf{x}$ is one such patch, with domain $D \subseteq \mathbf{E}^2$, then the composite

$$f(\mathbf{x}) : (u, v) \longmapsto \mathbf{x}(u, v) \longmapsto f(\mathbf{x}(u, v))$$

is a function from D to $\mathbf{R}$ in the sense discussed in *Part I.* We may, therefore, talk about the composite $f(\mathbf{x})$ being differentiable.

It is reasonable to *define* f to be differentiable if all the composites

$$f(\mathbf{x})$$

are differentiable for all the patches defining M. This is the motivation for the second paragraph of the section on page 143.

In practice it is quite usual to define a function on a surface by giving the composite $f(\mathbf{x})$. For example, suppose that M is the surface defined by the single patch

This does define a proper patch.

$$\mathbf{x}(u, v) = (u, v, uv), \quad u, v \in \mathbf{R}.$$

To define a function f on M we must specify the value that f takes at each point of M. One way to do this is in terms of u and v, since each point of M is of the form $\mathbf{x}(u, v)$. Thus we can define, say,

$$f(\mathbf{x}(u, v)) = u^2 + v^2 + u^2v^2.$$

This definition of f is the same as defining

$$f(\mathbf{p}) = p_1^2 + p_2^2 + p_3^2, \quad \mathbf{p} \in M$$

or

$$f = x^2 + y^2 + z^2, \text{ domain of } f \text{ is } M.$$

The definition of f in terms of the composite makes two things clear: firstly that the domain is M and secondly that the composite is differentiable. This is one of the reasons why failing to distinguish between f and $f(\mathbf{x})$ is not too terrible an abuse.

Note: This definition poses a problem for checking that such functions are differentiable; it is not enough to show that $f(\mathbf{x})$ is differentiable for the patch(es) used to define M. Differentiability must be shown for all conceivable patches in M.

This is a serious drawback and is remedied by the rather technical lemmas in the text. The end result is, reasonably enough, that the check needs to be made only for enough patches to cover the surface. (Thus in the above example, one patch is enough.)

We now turn to the other case: where the surface is the codomain. The situation is the following. We have a surface M, a (proper) patch $\mathbf{x}$ and a function F from, say, $\mathbf{E}^3$ to M

$$D \xrightarrow{\mathbf{x}} M \xleftarrow{F} \mathbf{E}^3.$$

Following the same strategy as in the first case, we need a composite which is an ordinary mapping in the sense of *Part I.* The mapping diagram above gives a clue as to what to do. We make use of the fact that $\mathbf{x}$ is a proper patch to invert it to give

$$D \xleftarrow{\mathbf{x}^{-1}} M \xleftarrow{F} \mathbf{E}^3$$

or, more conventionally,

$$\mathbf{E}^3 \xrightarrow{F} M \xrightarrow{\mathbf{x}^{-1}} D.$$

This leads to the definition in the third paragraph of the section in *O'Neill*, defining F to be differentiable if all such composites

$$\mathbf{x}^{-1}(F)$$

are differentiable as mappings.

The lemmas in *O'Neill* reduce the need to check all patches to checking just enough to cover the surface.

Proofs and results The proofs of 3.1–3.4 are not important, the results are.

The result that we shall make most use of is Lemma 3.1 which describes exactly what curves in a surface look like.

Tangent vectors Suppose that we have a surface M in $\mathbf{E}^3$ and a point $\mathbf{p}$ of M. Since $\mathbf{p}$ is a point of $\mathbf{E}^3$, it has its collection of tangent vectors—its tangent space. Some of these tangent vectors are of particular interest. Firstly those that are tangent to M (in the intuitive sense), secondly those that are normal to M.

Definition 3.5 formalizes the intuitive idea of being tangent to M by defining a tangent vector to be a tangent to M if it is the velocity vector of some curve in M. This definition is close to intuition but not very efficient as a practical test. Lemma 3.6 provides the practical test.

Note how the chain rule plays a central role in the proof of Lemma 3.6. There are some notes on the derivation of this form of the chain rule later in the commentary.

There is a passing remark in the proof of Lemma 3.6 that is worth careful note: the partial velocities form a basis for the tangent plane at each point of $\mathbf{x}(D)$. In a lot of work with surfaces they are the 'natural' choice of basis for the tangent plane because they are tightly linked to the definition of the surface via its patches. Calculations are usually best carried out in terms of the partial velocities. We hope that this general principle will be reinforced by the examples that you will meet.

There is an application of the chain rule used in the proof of Lemma 3.6 that is worth noting for its multiple uses.

> If $\mathbf{x}$ is a parametrization in a surface M and if
>
> $$\alpha(t) = \mathbf{x}(\alpha_1(t), \alpha_2(t))$$
>
> is a curve in the surface M, then
>
> $$\alpha' = \mathbf{x}_u(\alpha_1, \alpha_2)\alpha_1' + \mathbf{x}_v(\alpha_1, \alpha_2)\alpha_2'.$$

This result can be used to find curves that pass through a specified point with a specified velocity at that point.

Vector fields Definition 3.7 is much as you might expect. The reason for paying attention to *normal* vector fields will emerge in *Part V* of the course.

Lemma 3.8 gives one method of generating a normal vector: the gradient. From remarks earlier about partial velocities, it is tempting to use

$$\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v),$$

which is always normal to the surface, to attempt to define a normal vector field. This will not work for reasons that we explore in Section 6 of this text.

The chain rule The situation described in Corollary 3.4 arises frequently in our subsequent work. Although *O'Neill* does cover the corresponding version of the chain rule in an exercise, we think it important enough to deal with here.

The general situation is the following: we have two mappings

$$\mathbf{x} : D_1 \longrightarrow \mathbf{E}^3,$$
$$\mathbf{y} : D_2 \longrightarrow \mathbf{E}^3.$$

We also have two functions

$$\overline{u}, \overline{v} : D_1 \longrightarrow D_2.$$

The problem is to relate the partial velocities of $\mathbf{y}$ with those of $\mathbf{x}$.

We have dealt with exactly this situation in our work on mappings in *Part I.* If we define a mapping F from D_1 to D_2 by

$$F : (u,v) \longmapsto (\overline{u}(u,v), \overline{v}(u,v)),$$

then we have

$$\mathbf{y} = \mathbf{x}(F).$$

It follows, by the composite rule for derivative maps, that

$$\mathbf{y}_* = \mathbf{x}_*(F)F_*.$$

The chain rule that we are aiming for is obtained by expressing this last equation in matrix form. To do so we need to use the coordinate functions of $\mathbf{x}$ and $\mathbf{y}$.

Suppose that

$$\mathbf{x} = (f_1, f_2, f_3),$$
$$\mathbf{y} = (g_1, g_2, g_3).$$

Then the Jacobian matrices for $\mathbf{x}_*$ and $\mathbf{y}_*$ are

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\[2ex] \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \\[2ex] \dfrac{\partial f_3}{\partial u} & \dfrac{\partial f_3}{\partial v} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \dfrac{\partial g_1}{\partial u} & \dfrac{\partial g_1}{\partial v} \\[2ex] \dfrac{\partial g_2}{\partial u} & \dfrac{\partial g_2}{\partial v} \\[2ex] \dfrac{\partial g_3}{\partial u} & \dfrac{\partial g_3}{\partial v} \end{pmatrix}.$$

(Note that the columns are the partial velocities.) The Jacobian matrix for F_* is

$$\begin{pmatrix} \dfrac{\partial \overline{u}}{\partial u} & \dfrac{\partial \overline{u}}{\partial v} \\[2ex] \dfrac{\partial \overline{v}}{\partial u} & \dfrac{\partial \overline{v}}{\partial v} \end{pmatrix}.$$

Applying the composite rule gives

$$\begin{pmatrix} \dfrac{\partial g_1}{\partial u} & \dfrac{\partial g_1}{\partial v} \\[2ex] \dfrac{\partial g_2}{\partial u} & \dfrac{\partial g_2}{\partial v} \\[2ex] \dfrac{\partial g_3}{\partial u} & \dfrac{\partial g_3}{\partial v} \end{pmatrix}(u,v) = \begin{pmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\[2ex] \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \\[2ex] \dfrac{\partial f_3}{\partial u} & \dfrac{\partial f_3}{\partial v} \end{pmatrix}(\overline{u},\overline{v}) \begin{pmatrix} \dfrac{\partial \overline{u}}{\partial u} & \dfrac{\partial \overline{u}}{\partial v} \\[2ex] \dfrac{\partial \overline{v}}{\partial u} & \dfrac{\partial \overline{v}}{\partial v} \end{pmatrix}(u,v).$$

The first column of the product is

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial u}(\overline{u},\overline{v})\dfrac{\partial \overline{u}}{\partial u} + \dfrac{\partial f_1}{\partial v}(\overline{u},\overline{v})\dfrac{\partial \overline{v}}{\partial u} \\ \dfrac{\partial f_2}{\partial u}(\overline{u},\overline{v})\dfrac{\partial \overline{u}}{\partial u} + \dfrac{\partial f_2}{\partial v}(\overline{u},\overline{v})\dfrac{\partial \overline{v}}{\partial u} \\ \dfrac{\partial f_3}{\partial u}(\overline{u},\overline{v})\dfrac{\partial \overline{u}}{\partial u} + \dfrac{\partial f_3}{\partial v}(\overline{u},\overline{v})\dfrac{\partial \overline{v}}{\partial u} \end{pmatrix}.$$

Since

$$\mathbf{x}_u = \left(\frac{\partial f_1}{\partial u}, \frac{\partial f_2}{\partial u}, \frac{\partial f_3}{\partial u}\right)$$

etc., comparing the first columns of the composite rule equation gives

$$\mathbf{y}_u = \mathbf{x}_u(\overline{u},\overline{v})\frac{\partial \overline{u}}{\partial u} + \mathbf{x}_v(\overline{u},\overline{v})\,\frac{\partial \overline{v}}{\partial u}.$$

Similarly, the second columns give

$$\mathbf{y}_v = \mathbf{x}_u(\overline{u},\overline{v})\frac{\partial \overline{u}}{\partial v} + \mathbf{x}_v(\overline{u},\overline{v})\,\frac{\partial \overline{v}}{\partial v}.$$

Note how both partial velocities of $\mathbf{x}$ appear in each of the partial velocities for $\mathbf{y}$.

This form of the chain rule will be used frequently in our subsequent work.

Directional derivatives The final definition in the section is of great importance. It is worth noting that something that was a lemma in $\mathbf{E}^3$ (Lemma 4.6 of *Part I*) has become the *definition* for surfaces. This strategy of using a result as the basis of a more general definition will be used again in *Part VI.*

There are consequences of Definition 3.10 that *O'Neill* deals with in Exercise 4 that we consider of sufficient importance to discuss here. (If you wish you could try Exercise 4 on page 149 before reading what follows.)

We remarked above that it is often useful to work in terms of the partial velocities, since they form a basis for the tangent plane at each point. Thus it would be convenient to have a simple, general method of calculating

$$\mathbf{x}_u[f] \quad \text{and} \quad \mathbf{x}_v[f]$$

for a real-valued function f on a surface.

To be specific, we shall find

$$\mathbf{v}_\mathbf{p}[f],$$

where

$$\mathbf{v} = \mathbf{x}_u(u_0, v_0) \quad \text{at } \mathbf{p} = \mathbf{x}(u_0, v_0).$$

The 'obvious' curve through $\mathbf{x}(u_0, v_0)$ with velocity $\mathbf{x}_u(u_0, v_0)$ is the u-parameter curve

$$\alpha(t) = \mathbf{x}(u_0 + t, v_0).$$

Applying the definition

$$\begin{aligned} \mathbf{v}_\mathbf{p}[f] &= \left(\frac{d}{dt}f(\alpha(t))\right)(0) \\ &= \left(\frac{\partial}{\partial u}f(\alpha(t))\frac{d(u_0+t)}{dt} + \frac{\partial}{\partial v}f(\alpha(t))\frac{dv_0}{dt}\right)(0) \quad \text{(chain rule)} \\ &= \left(\frac{\partial}{\partial u}f(\alpha(t))\right)(0) \\ &= \frac{\partial}{\partial u}(f(\mathbf{x}))(u_0, v_0)). \end{aligned}$$

Generalizing, we have

$$\mathbf{x}_u[f] = \frac{\partial f(\mathbf{x})}{\partial u}.$$

Similar calculations give the corresponding result for $\mathbf{x}_v$:

$$\mathbf{x}_v[f] = \frac{\partial f(\mathbf{x})}{\partial v}.$$

Example Suppose that M is the surface

$$M : z = xy$$

parametrized by

$$\mathbf{x}(u, v) = (u, v, uv).$$

Let f be the function giving the distance of a point on M from the origin, that is, f is the restriction to M of

$$\sqrt{x^2 + y^2 + z^2}.$$

Then

$$\begin{aligned}\mathbf{x}_u(u, v)[f] &= \frac{\partial f(\mathbf{x})}{\partial u} \\ &= \frac{\partial}{\partial u}(x^2 + y^2 + z^2)(u, v, uv) \\ &= \frac{\partial}{\partial u}\sqrt{u^2 + v^2 + u^2v^2} \\ &= \frac{1}{2}\frac{1}{\sqrt{u^2 + v^2 + u^2v^2}}(2u + 2uv^2) \\ &= \frac{u(1 + v^2)}{\sqrt{u^2 + v^2 + u^2v^2}}.\end{aligned}$$

Similarly

$$\mathbf{x}_v(u, v)[f] = \frac{v(1 + u^2)}{\sqrt{u^2 + v^2 + u^2v^2}}.$$

You may recall from *Part I* that we observed that directional differentiation (of various kinds) with respect to the natural frame field always reduced to partial differentiation with respect to x, y and z. The results above indicate that, on surfaces, we have a corresponding simplicity if we work with respect to partial velocities. All forms of directional differentiation with respect to the partial velocities will reduce to

$$\frac{\partial}{\partial u} \quad \text{and} \quad \frac{\partial}{\partial v}.$$

The calculations above actually constitute a proof of this observation since covariant differentiation etc. are all based, ultimately, on the definition of

$$\mathbf{v}_{\mathbf{p}}[f].$$

■

In what follows we shall make heavy use of directional differentiation with respect to partial velocities, together with the chain rule.

Exercise 3.1 *O'Neill*, page 149, Exercise 1. The patch referred to is

$$\mathbf{x}(u, v) = (r\cos v\cos u, r\cos v\sin u, r\sin v), \quad -\pi < u < \pi, \quad -\frac{\pi}{2} < v < \frac{\pi}{2}.$$

Exercise 3.2 *O'Neill*, page 149, Exercise 2. The parametrization referred to is

$$\mathbf{x}(u, v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u), \quad 0 < r < R.$$

Exercise 3.3 *O'Neill*, page 150, Exercise 5.

Exercise 3.4 *O'Neill*, page 150, Exercise 8(a) and (b).

Exercise 3.5 Let M be the surface defined by the single patch $\mathbf{x}$, where

$$\mathbf{x}(u,v) = (u, v, uv), \quad (u,v) \in \mathbf{E}^2.$$

Let f be the function defined on M by

$$f(\mathbf{p}) = p_1^2 + p_2^2 - p_3.$$

Find $\mathbf{x}_u[f]$ and $\mathbf{x}_v[f]$.

Exercise 3.6 Let $\mathbf{x}$ and $\mathbf{y}$ be mappings from $\mathbf{E}^2$ to $\mathbf{E}^3$ defined as follows:

$$\begin{aligned} \mathbf{x}(u,v) &= (u, v, uv), \\ \mathbf{y}(u,v) &= \mathbf{x}(u^2+v^2, u^2-v^2). \end{aligned}$$

Use both the chain rule and direct calculation to find $\mathbf{y}_u(u,v)$ and $\mathbf{y}_v(u,v)$ and check that your answers are the same by both methods.

[*Solutions on page 36*]

4 Differential forms on a surface

Read *O'Neill: Chapter IV, Section 4, pages 152–157.*

Erratum *O'Neill*, page 157, in Exercise 5, the first displayed line

for $\mathbf{v}_\mathbf{p}[g(f)] = g'(f)\mathbf{v}_\mathbf{p}[f]$. read $\mathbf{v}_\mathbf{p}[g(f)] = g'(f(\mathbf{p}))\mathbf{v}_\mathbf{p}[f]$. ■

In this section we continue the process (begun in the previous section) of extending the definitions of *Part I* to surfaces.

1-forms Since we have defined tangent vectors on a surface, the *Part I* definition of 1-form carries over directly to surfaces.

Examples of 1-forms As with $\mathbf{E}^3$, one way of defining 1-forms is as differentials of real-valued functions. The definition of differential is just as for $\mathbf{E}^3$:

$$df(\mathbf{v}_\mathbf{p}) = \mathbf{v}_\mathbf{p}[f].$$

The proof that df is a 1-form is exactly the same as the one given in *Part I.*

Two special cases are of particular interest. For a surface M and a patch $\mathbf{x}$ in M, we define two functions as follows:

$$\tilde{u} : \mathbf{x}(u,v) \longmapsto u \quad \text{and} \quad \tilde{v} : \mathbf{x}(u,v) \longmapsto v.$$

These *are* functions because patches are one–one.

Next we calculate the values of $d\tilde{u}$ and $d\tilde{v}$ on $\mathbf{x}_u$ and $\mathbf{x}_v$.

$$\begin{aligned} d\tilde{u}(\mathbf{x}_u) &= \mathbf{x}_u[\tilde{u}] \\ &= \frac{\partial \tilde{u}(\mathbf{x}(u,v))}{\partial u} \\ &= \frac{\partial u}{\partial u} \\ &= 1. \end{aligned}$$

$$\begin{aligned} d\tilde{u}(\mathbf{x}_v) &= \mathbf{x}_v[\tilde{u}] \\ &= \frac{\partial \tilde{u}(\mathbf{x}(u,v))}{\partial v} \\ &= \frac{\partial u}{\partial v} \\ &= 0. \end{aligned}$$

$$\begin{aligned} d\tilde{v}(\mathbf{x}_u) &= \mathbf{x}_u[\tilde{v}] \\ &= \frac{\partial \tilde{v}(\mathbf{x}(u,v))}{\partial u} \\ &= \frac{\partial v}{\partial u} \\ &= 0. \end{aligned}$$

$$\begin{aligned} d\tilde{v}(\mathbf{x}_v) &= \mathbf{x}_v[\tilde{v}] \\ &= \frac{\partial \tilde{v}(\mathbf{x}(u,v))}{\partial v} \\ &= \frac{\partial v}{\partial v} \\ &= 1. \end{aligned}$$

These may remind you of the results that we obtained for $\mathbf{E}^3$:

$$dx_i(U_j) = \delta_{ij}.$$

From the above results we can calculate the values of $d\tilde{u}$ and $d\tilde{v}$ on any vector $\mathbf{v_p}$ tangent to M. We express $\mathbf{v_p}$ in terms of the partial velocities as, say,

$$\mathbf{v_p} = a\mathbf{x}_u(u,v) + b\mathbf{x}_v(u,v)$$

and then use linearity:

$$\begin{aligned} d\tilde{u}(\mathbf{v_p}) &= d\tilde{u}(a\mathbf{x}_u(u,v) + b\mathbf{x}_v(u,v)) \\ &= ad\tilde{u}(\mathbf{x}_u(u,v)) + bd\tilde{v}(\mathbf{x}_v(u,v)) \\ &= a \times 1 + b \times 0 \\ &= a. \end{aligned}$$

Similar calculations show that

$$d\tilde{v}(\mathbf{v_p}) = b.$$

Thus, just as dx, dy and dz pick out the Euclidean components of a tangent vector in $\mathbf{E}^3$, so $d\tilde{u}$ and $d\tilde{v}$ pick out the $\mathbf{x}_u$ and $\mathbf{x}_v$ components of a vector tangent to a surface.

Abuse of notation In spite of the fact that

$$u \quad \text{and} \quad \tilde{u}$$

are different functions (they have different domains), it is usual to call both of them u. With this abused notation we write

$u : (u,v) \longmapsto u$
$\tilde{u} : \mathbf{x}(u,v) \longmapsto u.$

$$\begin{aligned} du(\mathbf{x}_u) &= 1, \\ du(\mathbf{x}_v) &= 0, \\ dv(\mathbf{x}_u) &= 0, \\ dv(\mathbf{x}_v) &= 1. \end{aligned}$$

2-forms Definition 4.1 is the formal statement of the idea that was mentioned briefly in the commentary on differential forms in *Part I.*

Just for reference, we spell out the linearity conditions.

$$\begin{aligned} \eta(a\mathbf{v}_1 + b\mathbf{v}_2, \mathbf{w}) &= a\eta(\mathbf{v}_1, \mathbf{w}) + b\eta(\mathbf{v}_2, \mathbf{w}), \\ \eta(\mathbf{v}, a\mathbf{w}_1 + b\mathbf{w}_2) &= a\eta(\mathbf{v}, \mathbf{w}_1) + b\eta(\mathbf{v}, \mathbf{w}_2). \end{aligned}$$

Please note carefully the remark just after the proof of Lemma 4.2. This implies that once we know the value of a 2-form on a basis at each point of the surface, then we know all about the 2-form. We shall exploit this to define 2-forms by giving their effect on the basis consisting of the partial velocities. That is we shall often define a 2-form, η say, by giving

$$\eta(\mathbf{x}_u, \mathbf{x}_v).$$

Lemma 4.2 then shows how to calculate the value of η on other tangent vectors. If we have two tangent vectors at a point defined by

$$\mathbf{v} = a\mathbf{x}_u + b\mathbf{x}_v \quad \text{and} \quad \mathbf{w} = c\mathbf{x}_u + d\mathbf{x}_v,$$

then we apply Lemma 4.2 to obtain

$$\eta(\mathbf{v}, \mathbf{w}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \eta(\mathbf{x}_u, \mathbf{x}_v) = (ad - bc)\eta(\mathbf{x}_u, \mathbf{x}_v).$$

Wedge product Definition 4.3 links the formal algebra of wedge products that we did in *Part I* with Definition 4.1.

Example We have defined the 1-forms du and dv above. These enable us to define the 2-form $du \wedge dv$. We can calculate the value of $du \wedge dv$ on $(\mathbf{x}_u, \mathbf{x}_v)$ as follows:

We shall, in future, denote this wedge product by $du\, dv$.

$$\begin{aligned}(du \wedge dv)(\mathbf{x}_u, \mathbf{x}_v) &= du(\mathbf{x}_u)\, dv(\mathbf{x}_v) - du(\mathbf{x}_v)\, dv(\mathbf{x}_u) \\ &= 1 \times 1 - 0 \times 0 \\ &= 1.\end{aligned}$$

It follows, from Lemma 4.2, that

$$(du \wedge dv)(a\mathbf{x}_u + b\mathbf{x}_v, c\mathbf{x}_u + d\mathbf{x}_v) = ad - bc.$$ ■

Exterior derivatives Our eventual aim is to use the connection forms and structural equations to obtain geometric information about surfaces. The structural equations involve exterior derivatives of 1-forms, so these need to be defined for surfaces. As has been the case so far, the definition is given in terms of patches. This is the content of Definition 4.4.

Because patches may overlap, Lemma 4.5 is needed to ensure that there is no conflict in regions of overlap. The reassurance provided by the result is important, the details of the proof are not. You may wish to note that the proof rests on several uses of the chain rule and the fact that

$$\frac{\partial}{\partial u}\frac{\partial f}{\partial v} = \frac{\partial}{\partial v}\frac{\partial f}{\partial u}.$$

The exterior derivative obeys the usual linearity and Leibniz properties. (*O'Neill* relegates the proofs to an exercise, but you are free to assume them.)

Computational methods It is convenient to be able to work with differential forms in terms of du and dv.

Compare this with $\mathbf{E}^3$, where we worked in terms of dx etc.

We now discuss some results which generalize the remarks made in *O'Neill* under Example 4.7.

Firstly, suppose that we are given a 1-form ϕ on a surface M and that $\mathbf{x}$ is a patch in M. Then, ϕ is completely determined (in the image of $\mathbf{x}$) by its values on $\mathbf{x}_u$ and $\mathbf{x}_v$. Define the functions f and g by

$$\begin{aligned}f(\mathbf{x}(u,v)) &= \phi(\mathbf{x}_u(u,v)), \\ g(\mathbf{x}(u,v)) &= \phi(\mathbf{x}_v(u,v)).\end{aligned}$$

Then

$$\begin{aligned}\phi(\mathbf{x}_u(u,v)) &= f(\mathbf{x}(u,v))\, du(\mathbf{x}_u(u,v)) + g(\mathbf{x}(u,v))\, dv(\mathbf{x}_u(u,v)), \\ \phi(\mathbf{x}_v(u,v)) &= f(\mathbf{x}(u,v))\, du(\mathbf{x}_v(u,v)) + g(\mathbf{x}(u,v))\, dv(\mathbf{x}_v(u,v)).\end{aligned}$$

Note: The second term in the first line and the first term in the second line are zero.

Writing these two equations in 'functional' form shows that

$$\phi = f\, du + g\, dv.$$

Thus any 1-form on M can be expressed as a linear combination of du and dv.

If more than one patch is needed to cover M then the linear combination may change from image of patch to image of patch. However, the various lemmas proved in *O'Neill* guarantee that all will agree on any overlaps.

We can now express the exterior derivative of a 1-form

$$\phi = f\, du + g\, dv$$

in terms of the wedge product $du\,dv$. Applying Definition 4.4 to ϕ gives

$$\begin{aligned}
d\phi(\mathbf{x}_u,\mathbf{x}_v) &= \frac{\partial}{\partial u}\phi(\mathbf{x}_v) - \frac{\partial}{\partial v}\phi(\mathbf{x}_u)\\
&= \frac{\partial}{\partial u}(f(\mathbf{x})\,du(\mathbf{x}_v) + g(\mathbf{x})\,dv(\mathbf{x}_v)) - \frac{\partial}{\partial v}(f(\mathbf{x})\,du(\mathbf{x}_u) + g(\mathbf{x})\,dv(\mathbf{x}_u))\\
&= \frac{\partial}{\partial u}(0 + g(\mathbf{x})) - \frac{\partial}{\partial v}(f(\mathbf{x}) + 0)\\
&= \frac{\partial g(\mathbf{x})}{\partial u} - \frac{\partial f(\mathbf{x})}{\partial v}.
\end{aligned}$$

However, we showed earlier that

$$(du\,dv)(\mathbf{x}_u,\mathbf{x}_v) = 1,$$

so

$$d\phi(\mathbf{x}_u,\mathbf{x}_v) = \left(\frac{\partial g(\mathbf{x})}{\partial u} - \frac{\partial f(\mathbf{x})}{\partial v}\right)(du\,dv)(\mathbf{x}_u,\mathbf{x}_v).$$

We can write this in functional form as

$$d\phi = \left(\frac{\partial g(\mathbf{x})}{\partial u} - \frac{\partial f(\mathbf{x})}{\partial v}\right)du\,dv.$$

If we calculated $d\phi$ as in *Part I*, we would proceed as follows

$$\begin{aligned}
d\phi &= d(f\,du + g\,dv)\\
&= df \wedge du + f\,d(du) + dg \wedge dv + g\,d(dv)\\
&\quad \text{(linearity and two applications of Leibniz)}\\
&= df\,du + dg\,dv\\
&= \left(\frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv\right)du + \left(\frac{\partial g}{\partial u}du + \frac{\partial g}{\partial v}dv\right)dv\\
&= \frac{\partial f}{\partial v}dv\,du + \frac{\partial g}{\partial u}du\,dv\\
&= \left(\frac{\partial g}{\partial u} - \frac{\partial f}{\partial v}\right)du\,dv.
\end{aligned}$$

The second set of calculations above involve the same abuse of notation as earlier when we ignored the distinction between u and $\tilde{u}$. Since f and g are defined on M, for any patch $\mathbf{x}$ we automatically have the composites

$$f(\mathbf{x}), g(\mathbf{x}) : D \longrightarrow \mathbf{R},$$

where D is the domain of $\mathbf{x}$. As we have indicated, we are far from scrupulous about distinguishing between f and the composite $f(\mathbf{x})$.

Parametrizations In practice, we often work with parametrizations of a surface rather than collections of patches. For example, earlier we worked with the parametrization

Example 2.6 of Section 2.

$$\mathbf{x}(u,v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u),\ R > r > 0, \quad (u,v) \in \mathbf{E}^2$$

of the torus of revolution T. Now $\mathbf{x}$ is not a patch, since it is not one–one. However, by using exactly the same *rule* but with a restricted domain, we can create a patch from $\mathbf{x}$ which covers part of T. Thus, choosing a domain D_1

$$D_1 = \{(u,v) \in \mathbf{E}^2 : 0 < u < \pi/2, 0 < v < \pi/2\}$$

gives a patch $\mathbf{x}_1$ which covers part of T.

If we choose a collection of domains, on each of which $\mathbf{x}$ is one–one, then we can produce enough patches to cover T.

Now suppose that we have a function from T to $\mathbf{R}$ defined by

$$f(\mathbf{p}) = p_1^2 + p_2^2 - p_3^2, \quad \mathbf{p} \in T.$$

On *any* of the patches obtained from $\mathbf{x}$, the rule for the composite $f(\mathbf{x})$ will be, after simplification,

$$f(\mathbf{x}) = (R + r\cos u)^2 - r^2 \sin^2 u.$$

Thus any calculations using the composites will look the same for all patches and lemmas such as Lemma 4.5 ensure that there is no conflict on overlaps between patches.

We can say, for example, that

$$df = (-2r(R + r\cos u)\sin u - 2r^2 \sin u \cos u)du + 0\,dv$$

working straight from the parametrization rather than with individual patches. All we need to know is that we *could* define a covering set of patches from the parametrization if we wanted to.

Generally speaking it is more convenient to work with a single parametrization for a surface than with a collection of patches. Most of the examples of surfaces in this course will have single parametrizations available.

A single parametrization may not exist.

Exercise 4.1 *O'Neill*, page 157, Exercise 1.

Exercise 4.2 Let M be the surface parametrized by the mapping

$$\mathbf{x}(u, v) = (u, v, uv), \quad (u, v) \in \mathbf{E}^2.$$

(Note that $\mathbf{x}$ is actually a patch.)

Let ϕ and ψ be the 1-forms defined on M as follows. If $\mathbf{v_p}$ is a vector tangent to M at $\mathbf{p} = \mathbf{x}(u, v)$, then

$$\phi(\mathbf{v_p}) = \mathbf{v} \cdot \mathbf{x}_u(u, v),$$
$$\psi(\mathbf{v_p}) = \mathbf{v} \cdot \mathbf{x}_v(u, v).$$

(a) Show that $\mathbf{p} = (-1, 3, -3)$ is in M and that

$$\mathbf{v_p} = (1, 2, 1)_\mathbf{p}$$

is tangent to M.

(b) Find $\phi(\mathbf{v_p})$ and $\psi(\mathbf{v_p})$, where $\mathbf{v_p}$ is as above.

(c) Find the effect of ϕ and ψ on $\mathbf{x}_u$ and $\mathbf{x}_v$.

(d) Assume that

$$\phi = a\,du + b\,dv$$

for some functions a and b. By using the results of the previous part, find a and b.

(e) Express ψ as a linear combination of du and dv.

Exercise 4.3 This exercise continues the previous one and uses the same definitions and notation.

Calculate $\phi \wedge \psi$, expressing your answer as a multiple of $du\,dv$. Hence calculate

$$(\phi \wedge \psi)(\mathbf{x}_u, \mathbf{x}_v).$$

Check this last result by using Definition 4.3 directly.

[*Solutions on page 38*]

5 Mappings of surfaces

Read *O'Neill: Chapter IV, Section 5, pages 158–164.*

Note: Theorem 5.4 and Example 5.5 will not be of any real importance in this course.

We now turn to the problem of defining mappings between surfaces. Previously a mapping has simply been a differentiable function between Euclidean spaces. There is no difficulty about the idea of a *function* from one surface to another. What requires a little care is the definition of differentiability.

Since two surfaces, M and N say, must each be defined by collections of patches, we transfer the definition of differentiability to the domains of the patches. Fig. 4.29 and Definition 5.1 do this.

This is the third application of 'transfer to domains of patches' to obtain definitions. If we summarize the definitions of differentiability in the three cases, the family resemblance may be clearer.

Firstly, for a function $f : M \longrightarrow \mathbf{R}$, we have

$$D \xrightarrow{\mathbf{x}} M \xrightarrow{f} \mathbf{R}$$

and differentiability of f is defined in terms of differentiability of the composite

$$f\mathbf{x}.$$

Secondly, for a function $F : \mathbf{E}^3 \longrightarrow M$, we have

$$\mathbf{E}^3 \xrightarrow{F} M \xrightarrow{\mathbf{x}^{-1}} D$$

and differentiability is defined in terms of differentiability of the composite

$$\mathbf{x}^{-1}F.$$

The latest definition is a combination of these two. For a function $F : M \longrightarrow N$, we have

$$D \xrightarrow{\mathbf{x}} M \xrightarrow{F} N \xrightarrow{\mathbf{y}^{-1}} E$$

and differentiability is defined in terms of the differentiability of the composite

$$\mathbf{y}^{-1}F\mathbf{x}.$$

Having made the definition, there is the usual remark that it is sufficient to do the check for enough patches to cover both surfaces.

Example 5.2 This example illustrates the fact that we usually work with parametrizations (rather than patches), in the knowledge that we *could* set up patches by using a suitable collection of domains.

The rule, $\mathbf{x}$, given for the geographical patch needs a domain restriction to prevent it being periodic. However, *O'Neill* acknowledges the problem in a comment and then ignores it!

Probably the most important feature of this example is that it shows how mappings between surfaces are usually specified when we are working in $\mathbf{E}^3$. The mapping F is, effectively, defined by

$$F(\mathbf{x}(u,v)) = \mathbf{y}(u, \sin v).$$

What we have done is to define the parameters of the image in terms of the parameters of the point in the domain. That is, F is specified by giving the rule

$$(u,v) \longmapsto (u, \sin v)$$

for the composite

$$\mathbf{y}^{-1}F\mathbf{x}.$$

Uses We shall have one main application of mappings between surfaces. We shall consider a particular type of map between a surface M and the unit sphere. Since the unit sphere is, in some sense, a surface with a 'standard' shape, the amount of distortion needed to map M onto the unit sphere will give a measure of the 'shape' of M

These ideas will be made precise in *Part V*.

Derivative maps As in the case of mappings from $\mathbf{E}^m$ to $\mathbf{E}^n$, we next discuss what mappings do to tangent vectors.

Definition 5.3 is a generalization of the definition of

$$\mathbf{v}_\mathbf{p}[f]$$

for a function f from M to $\mathbf{R}$. It is made in terms of curves in the surfaces.

Examples Whenever a definition like 5.3 is made, it is a good idea to apply it to the partial velocities.

This is because of the central role played by the partial velocities in providing bases for the tangent planes.

The partial velocities are the velocities of the parameter curves. It follows from the definition that

$$F_*(\mathbf{x}_u)$$

is the velocity of the image in N of the u-parameter curve in M, that is,

$$F_*(\mathbf{x}_u(u,v)) = \frac{\partial}{\partial u}F(\mathbf{x}(u,v)).$$

The partial derivative can be calculated by using the chain rule that we have discussed previously.

As a concrete example, we shall pursue Example 5.2 of *O'Neill* a little further. There we have

$$F(\mathbf{x}(u,v)) = \mathbf{y}(u, \sin v).$$

Hence,

$$\begin{aligned} F_*(\mathbf{x}_u(u,v)) &= \frac{\partial}{\partial u}F(\mathbf{x}(u,v)) \\ &= \frac{\partial}{\partial u}\mathbf{y}(u,\sin v) \\ &= \mathbf{y}_u(u,\sin v)\frac{\partial u}{\partial u} + \mathbf{y}_v(u,\sin v)\frac{\partial \sin v}{\partial u} \\ &= \mathbf{y}_u(u,\sin v). \end{aligned}$$

Similarly

Note the use of the chain rule.

$$\begin{aligned} F_*(\mathbf{x}_v(u,v)) &= \frac{\partial}{\partial v}F(\mathbf{x}(u,v)) \\ &= \frac{\partial}{\partial v}\mathbf{y}(u,\sin v) \\ &= \mathbf{y}_u(u,\sin v)\frac{\partial u}{\partial v} + \mathbf{y}_v(u,\sin v)\frac{\partial \sin v}{\partial v} \\ &= \cos v\, \mathbf{y}_v(u,\sin v). \end{aligned}$$

We can check these results directly. We have

$$F(\mathbf{x}(u,v)) = (\cos u, \sin u, \sin v),$$

$$\frac{\partial}{\partial u}F(\mathbf{x}(u,v)) = (-\sin u, \cos u, 0),$$
$$\frac{\partial}{\partial v}F(\mathbf{x}(u,v)) = (0, 0, \cos v).$$

But

$$\mathbf{y}_u(u,v) = (-\sin u, \cos u, 0),$$
$$\mathbf{y}_v(u,v) = (0,0,1).$$

So

$$\mathbf{y}_u(u,\sin v) = (-\sin u, \cos u, 0) = F_*(\mathbf{x}_u(u,v)),$$
$$\mathbf{y}_v(u,\sin v) = (0,0,1),$$

and so

$$\cos v\,\mathbf{y}_v(u,v) = (0,0,\cos v) = F_*(\mathbf{x}_v(u,v)).$$

Warning: The displayed line in the middle of page 161 of *O'Neill* is true but very misleading.

The problem is that the result displayed seems to be in direct contradiction to the result that we obtained in the above example:

$$F_*(x_v(u,v)) = \cos v\,\mathbf{y}_v(u,\sin v).$$

There is no contradiction if you inspect the first paragraph on page 161. The displayed result applies to only the *special case*

$$\mathbf{y}(u,v) = F(\mathbf{x}(u,v)).$$

This is the case where the corresponding map between the domains of $\mathbf{x}$ and $\mathbf{y}$ is the identity:

$$(u,v) \longmapsto (u,v).$$

In the example, we had

$$(u,v) \longmapsto (u,\sin v),$$

Hence the more complicated result.

We strongly advise you to ignore the special case and always work from

$$F_*(\mathbf{x}_u) = \frac{\partial}{\partial u}F(\mathbf{x}) \quad \text{and} \quad F_*(\mathbf{x}_v) = \frac{\partial}{\partial v}F(\mathbf{x}),$$

together with the chain rule, as we did in the last example.

We now know how to transfer points and tangent vectors from one surface to another: points are transferred using mappings, tangent vectors by using the associated derivative maps. We now turn to transfer of differential forms.

Definition 5.6 The transfer of forms defined here is in the *opposite direction* to the transfer of points and tangent vectors. The mapping F transfers points from M to N. Similarly, the derivative map transfers tangent vectors on M to tangent vectors on N. However, F^* uses forms on N to define forms on M.

O'Neill talks about 'pullbacks' without explanation. We refer to F^* as a pullback because it transfers information in the opposite direction to F, that is from codomain to domain.

Example We shall develop Example 5.2 from *O'Neill*. We have the following mapping

$$F : \Sigma \longrightarrow C, \quad \text{where } F(\mathbf{x}(u,v)) = \mathbf{y}(u,\sin v).$$

Now, like any surface, C has the 1-forms du and dv defined on it. As a reminder, they satisfy

$$du(a\mathbf{y}_u + b\mathbf{y}_v) = a,$$
$$dv(a\mathbf{y}_u + b\mathbf{y}_v) = b.$$

Let us find the 1-forms

$$F^*(du) \quad \text{and} \quad F^*(dv)$$

on Σ. To know all about a 1-form we need its value on a basis of tangent vectors at each point; the logical basis to use (as ever) consists of the partial velocities, $\mathbf{x}_u$ and $\mathbf{x}_v$.

We shall make use of the results that we have found earlier, namely

$$\begin{aligned}F_*(\mathbf{x}_u(u,v)) &= \mathbf{y}_u(u,\sin v),\\ F_*(\mathbf{x}_v(u,v)) &= \cos v\,\mathbf{y}_u(u,\sin v).\end{aligned}$$

Using Definition 5.6(1),

$$\begin{aligned}(F^*\,du)(\mathbf{x}_u(u,v)) &= du(F_*(\mathbf{x}_u(u,v)))\\ &= du(\mathbf{y}_u(u,\sin v))\\ &= 1.\end{aligned}$$

$$\begin{aligned}(F^*\,du)(\mathbf{x}_v(u,v)) &= du(F_*(\mathbf{x}_v(u,v)))\\ &= du(\cos u\,\mathbf{y}_v(u,\sin v))\\ &= 0.\end{aligned}$$

$$\begin{aligned}(F^*\,dv)(\mathbf{x}_u(u,v)) &= dv(F_*(\mathbf{x}_u(u,v)))\\ &= dv(\mathbf{y}_u(u,\sin v))\\ &= 0.\end{aligned}$$

$$\begin{aligned}(F^*\,dv)(\mathbf{x}_v(u,v)) &= dv(F_*(\mathbf{x}_v(u,v)))\\ &= dv(\cos u\,\mathbf{y}_v(u,\sin v))\\ &= \cos u.\end{aligned}$$

These four results completely define $F^*(du)$ and $F^*(dv)$. If we compare the four results with the values of du and dv on the partial velocities *on* Σ, we see that

$$F^*(du) = du \text{ and } F^*(dv) = \cos u\,dv.$$

To give an example of F_* acting on a 2-form, we shall calculate

$$F^*(du\,dv).$$

We have shown earlier that $(du\,dv)(\mathbf{y}_u, \mathbf{y}_v) = 1$.

Applying Definition 5.6(2) gives

$$\begin{aligned}(F^*(du,dv))(\mathbf{x}_u(u,v),\mathbf{x}_v(u,v)) &= (du\,dv)(F_*(\mathbf{x}_u(u,v)),F_*(\mathbf{x}_v(u,v)))\\ &= (du\,dv)(\mathbf{y}_u(u,\sin v),\cos u\,\mathbf{y}_v(u,\sin v))\\ &= \cos u(du\,dv)(\mathbf{y}_u(u,\sin v),\mathbf{y}_v(u,\sin v))\\ &= \cos u.\end{aligned}$$

As we noted earlier, a 2-form is completely determined by its effect on the pair of partial velocities. Since, on Σ,

$$(du\,dv)(\mathbf{x}_u(u,v),\mathbf{x}_v(u,v)) = 1,$$

we have that

$$(F^*(du,dv))(\mathbf{x}_u(u,v),\mathbf{x}_v(u,v)) = \cos u(du\,dv)(\mathbf{x}_u(u,v),\mathbf{x}_v(u,v)).$$

It follows that we may say

$$F^*(du\,dv) = \cos u\,du\,dv.$$

Theorem 5.7 If pulling back forms is to work properly, then we need the results of this theorem. Only one detail of the proof is really important. The strategy in the only case that *O'Neill* actually proves is as follows. The two forms

$$F^*(d\xi) \quad\text{and}\quad d(F^*(\xi))$$

are evaluated on a basis and the results shown to be equal. As usual, the basis chosen consists of the partial velocities. This is a good general strategy for showing that two 2-forms are equal. Their effects need be checked only for the pair

$$(\mathbf{x}_u, \mathbf{x}_v).$$

Exercise 5.1 (This is a continuation of Example 5.2.) Let Σ be the unit sphere and C be the cylinder parametrized by

$$\mathbf{x}(u,v) = (\cos v \cos u, \cos v \sin u, \sin v),$$
$$\mathbf{y}(u,v) = (\cos u, \sin u, v),$$

respectively, and let F be the mapping from Σ to C defined by

$$F(\mathbf{x}(u,v)) = \mathbf{y}(u, \sin v).$$

Let ϕ_1 and ϕ_2 be the 1-forms defined on C as follows. If $\mathbf{v_p}$ is a tangent vector to C at $\mathbf{p} = \mathbf{y}(u,v)$, then

$$\phi_1(\mathbf{v}) = \mathbf{v} \cdot \mathbf{y}_u(u,v),$$
$$\phi_2(\mathbf{v}) = \mathbf{v} \cdot \mathbf{y}_v(u,v).$$

(a) Calculate the values of ϕ_1 and ϕ_2 on the partial velocities of $\mathbf{y}$. Hence find the values of

$$F^*\phi_1 \text{ and } F^*\phi_2$$

on the partial velocities of $\mathbf{x}$.

(b) Calculate

$$(F^*(\phi_1 \wedge \phi_2))(\mathbf{x}_u(u,v), \mathbf{x}_v(u,v)).$$

Exercise 5.2 Consider the surface M defined by the single patch

$$\mathbf{x}(u,v) = (u, v, uv), \quad (u,v) \in \mathbf{E}^2$$

and the mapping F from M to M defined by

$$F(\mathbf{x}(u,v)) = \mathbf{x}(u+v, u-v).$$

The vector field U is defined on M by

$$U(\mathbf{x}(u,v)) = \frac{\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)}{||\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)||}.$$

The 2-form η is defined on M as follows. Suppose that $\mathbf{v_p}$ and $\mathbf{w_p}$ are tangent vectors to M at the point $\mathbf{p} = \mathbf{x}(u,v)$. Then

$$\eta(\mathbf{v}, \mathbf{w}) = \mathbf{v} \times \mathbf{w} \cdot U(\mathbf{p}).$$

Find

$$\eta(\mathbf{x}_u(u,v), \mathbf{x}_v(u,v))$$

and hence find

$$(F^*\eta)(\mathbf{x}_u(u,v), \mathbf{x}_v(u,v)).$$

[*Solutions on page 39*]

6 Topological properties

Read *O'Neill: Chapter IV, Section 7, pages 176–180.*

We need one definition from the (omitted) Section IV.6 of *O'Neill*.

Definition A 2-segment in a surface M is a differentiable mapping from a closed rectangle in $\mathbf{E}^2$ to M. That is, a mapping $\mathbf{x}$

$$\mathbf{x} : D \longrightarrow M,$$

where $D = \{(u,v) \in \mathbf{E}^2 : a \le u \le b,\ c \le v \le d\}$.

The ideas in this section are of great importance in the development of the subject beyond this course. However, we shall make rather limited use of them. The main one that we shall consider is orientability.

Orientability This is a global property: its definition demands that there should exist a 2-form that is non-zero *everywhere* on the surface. However, we can always break up a surface into bits which are orientable by using the patches that define the surface. The image of each patch has to be orientable, for reasons that we shall discuss shortly.

Normal vector fields There is an 'obvious' way to try to construct a normal vector field on a surface: use the cross product of partial velocities. Formally, for a patch $\mathbf{x}$ of the surface M, consider

$$\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v).$$

This is certainly normal to the surface at each point where it is defined. The problem of using this cross product to define a vector field on M arises on the overlap of patches. To see why, we have to recall some results from earlier in *O'Neill*.

If two patches $\mathbf{x}$ and $\mathbf{y}$ overlap in M, then, in the overlap, there are functions $\tilde{u}$ and $\tilde{v}$ such that

$$\mathbf{y}(u,v) = \mathbf{x}(\tilde{u},\tilde{v}).$$

Two applications of the chain rule give

$$\mathbf{y}_u(u,v) = \mathbf{x}_u(\tilde{u},\tilde{v})\frac{\partial\tilde{u}}{\partial u} + \mathbf{x}_v(\tilde{u},\tilde{v})\frac{\partial\tilde{v}}{\partial u},$$
$$\mathbf{y}_v(u,v) = \mathbf{x}_u(\tilde{u},\tilde{v})\frac{\partial\tilde{u}}{\partial v} + \mathbf{x}_v(\tilde{u},\tilde{v})\frac{\partial\tilde{v}}{\partial v}.$$

Careful algebra gives

$$\mathbf{y}_u(u,v) \times \mathbf{y}_v(u,v) = \left(\frac{\partial\tilde{u}}{\partial u}\frac{\partial\tilde{v}}{\partial v} - \frac{\partial\tilde{u}}{\partial v}\frac{\partial\tilde{v}}{\partial u}\right)\mathbf{x}_u(\tilde{u},\tilde{v}) \times \mathbf{x}_v(\tilde{u},\tilde{v}).$$

We cannot be sure that the factor in brackets is 1. Thus the 'definition' using the cross product is possibly inconsistent on overlaps.

What went wrong above is also the clue to the remedy: use the cross product to define a *unit* normal vector field. That is, define $U(\mathbf{x}(u,v))$ by

$$U(\mathbf{x}(u,v)) = \frac{\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)}{\|\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)\|}.$$

The fact that $\mathbf{x}$ is a patch, and hence one–one and regular, ensures that the norm is uniquely defined and never zero. The one–one property ensures that each point $\mathbf{p}$ of M arises from just one pair (u,v) of parameters. (See the exercise on Möbius bands below.)

Since the only difficulty we ran into with overlaps was with the size of the normal, not its direction, this modification solves the problem.

Since every surface can be covered by a collection of patches, we can define a unit normal vector field on the image of each patch by this method. The unit normal cannot vanish and so the image of each patch is orientable, even if the complete surface is not.

This method of constructing unit normal vector fields is of considerable use in *Part V*. It can be extended *carefully* from patches to parametrizations in cases where all choices of parameters for a given point give the same result.

Because of its relative unimportance *for this course*, we have set only one exercise in this section (on Möbius bands).

Exercise 6.1 (This exercise is based on *O'Neill*, pages 180–181, Exercise 7.) Let the curves β and δ be defined by

$$\begin{aligned}\beta(u) &= (\cos u, \sin u, 0),\\ \delta(u) &= \cos(u/2)\beta(u) + \sin(u/2)U_3.\end{aligned}$$

We define the Möbius band M to be the image of the parametrization

$$\mathbf{x}(u,v) = \beta(u) + v\delta(u),\ -\tfrac{1}{2} < v < \tfrac{1}{2}.$$

(a) Calculate the partial velocities and hence show that $\mathbf{x}$ is regular. You may use the results of Exercise 2, Section 2.

(b) The point $\mathbf{p} = (1,0,0)$ lies in M, since

$$\mathbf{p} = \mathbf{x}(0,0).$$

Show that it is possible to find a value u_0 of u with the following properties.

$$\begin{aligned}\mathbf{p} &= \mathbf{x}(u_0,0),\\ \mathbf{x}_u(u_0,0) &= \mathbf{x}_u(0,0),\\ \mathbf{x}_v(u_0,0) &= -\mathbf{x}_v(0,0).\end{aligned}$$

Now, suppose that M is orientable so that there exists a non-vanishing normal vector field Z on M.

Define a real-valued function f on M by

$$f(\mathbf{x}(u,v)) = \mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v) \cdot Z(\mathbf{x}(u,v)).$$

(c) Explain why f is differentiable, and hence continuous.

(d) Show that f must vanish somewhere on the parameter curve $v = 0$ and deduce that M is not orientable.

[*Solution on page 40*]

7 Summary

Read *O'Neill: Chapter IV, Section 9, pages 187–188.*

We have now extended most of the ideas from calculus on $\mathbf{E}^n$ to surfaces in $\mathbf{E}^3$. One or two definitions still need to be made, including that of covariant derivative. However, these will be based on the definition of

$$\mathbf{v}_\mathbf{p}[f] \quad \text{and} \quad V[f]$$

for surfaces. (Just as they were in $\mathbf{E}^n$.)

A number of useful principles have been established. The most useful are probably the following.

(a) Where possible, work in terms of the partial velocities.

(b) All forms of directional derivative with respect to the partial velocities reduce to partial differentiation with respect to the parameters.

(c) Showing equality of forms is best tackled by evaluating on the basis consisting of partial velocities.

The partial velocities of patches, and of parametrizations, have played a central role in this text and will continue to do so in the next.

8 Appendix: integration

Read *O'Neill: Chapter IV, Section 6, pages 167–173.*

This appendix is included in case you have been wondering about the link between differential forms and integration.

In the integrals that you have met, for example

M101, M203 and *MST204*

$$\int_0^1 2x\,dx,$$

the expression after the integral sign is what we are now calling a 1-form. More, since

$$2x\,dx = d(x^2),$$

this particular 1-form is the exterior derivative of a function (0-form).

We can express the evaluation of the above integral as follows:

$$\begin{aligned}\int_0^1 2x\,dx &= \int_0^1 d(x^2)\\ &= \left[x^2\right]_0^1\\ &= (x^2)(1) - (x^2)(0)\\ &= 1 - 0 = 1.\end{aligned}$$

This suggests that we could state the Fundamental Theorem of Calculus as follows.

M101 and *M203*

If ϕ is a 1-form on $\mathbf{R}$ and

$$\phi = df$$

for some function $f : \mathbf{R} \longrightarrow \mathbf{R}$, then

$$\int_a^b \phi = f(b) - f(a).$$

The definitions in this section of *O'Neill* seek to extend this definition in two ways. Firstly, to cases where ϕ is a 1-form on a surface. Secondly, to integration of 2-forms.

The first extension is the content of Definition 6.1. In the special case that the 1-form is the exterior derivative of a function we obtain Theorem 6.2 as an extension of the Fundamental Theorem of Calculus.

Definitions 6.3 and 6.4 and Theorem 6.5 provide the extension to 2-forms.

We shall not set any exercises on this section but the following worked example provides some motivation for one of the 2-forms that will be of concern later on.

Example In this example we shall consider the unit sphere Σ, parametrized by

$$\mathbf{x}_u(u, v) = (\cos u \cos v, \cos u \sin v, \sin u), \quad (u, v) \in \mathbf{E}^2.$$

We note that, by defining a suitable collection of subsets of $\mathbf{E}^2$, we could use $\mathbf{x}$ to create a covering of proper patches for Σ.

We now construct a unit normal vector field on Σ by using the partial velocities.

$$\begin{aligned}\mathbf{x}_u &= (-\sin u \cos v, -\sin u \sin v, \cos u),\\ \mathbf{x}_v &= (-\cos u \sin v, \cos u \cos v, 0),\\ \mathbf{x}_u \times \mathbf{x}_v &= (-\cos^2 u \cos v, -\cos^2 u \sin v, -\cos u \sin u(\cos^2 v + \sin^2 v))\\ &= -\cos u(\cos u \cos v, \cos u \sin v, \sin u).\\ \|\mathbf{x}_u \times \mathbf{x}_v\|^2 &= \cos^2 u(\cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \sin^2 u)\\ &= \cos^2 u.\end{aligned}$$

From the last two lines we can see that either of

$$\pm\frac{\mathbf{x}_u \times \mathbf{x}_v}{\cos u}$$

would define a unit normal. We choose to define

$$\begin{aligned} U(\mathbf{x}(u,v)) &= \frac{\mathbf{x}_u \times \mathbf{x}_v}{\cos u} \\ &= (-\cos u \cos v, -\cos u \sin v, -\sin u). \end{aligned}$$

Strictly, we should check that all choices of u and v for a particular point lead to a unique value of U. We shall not do so, although all is well.

Next, we use U to define a 2-form on Σ. If $\mathbf{v}$ and $\mathbf{w}$ are tangent vectors at $\mathbf{p} = \mathbf{x}(u,v)$, then we define the 2-form η by

$$\eta(\mathbf{v},\mathbf{w}) = \mathbf{v} \times \mathbf{w} \cdot U.$$

It is not difficult to check that η is, indeed, a 2-form.

If we evaluate η on the partial velocities, then we can express η in terms of $du\,dv$.

$$\begin{aligned} \eta(\mathbf{x}_u, \mathbf{x}_v) &= \mathbf{x}_u \times \mathbf{x}_v \cdot U \\ &= \cos u\, U \cdot U \quad \text{(from the definition of } U\text{)} \\ &= \cos u. \end{aligned}$$

The last step uses the fact that U is a *unit* vector field.

Comparing this result with

$$(du\,dv)(\mathbf{x}_u, \mathbf{x}_v) = 1,$$

we have

$$\eta = \cos u\, du\, dv.$$

Let us now try integrating this 2-form over the 'northern' hemisphere, H defined by

$$0 \le u \le \frac{\pi}{2} \quad \text{and} \quad 0 \le v \le 2\pi.$$

Theorem 6.5 can be applied provided that we can find a 1-form, ϕ, such that

$$\eta = d\phi.$$

Suppose that $\phi = a\,du + b\,dv$. Then, from first principles,

We could also have used Example 4.7 from *O'Neill*.

$$\begin{aligned} d\phi &= \left(\frac{\partial a}{\partial u}du + \frac{\partial a}{\partial v}dv\right) du + \left(\frac{\partial b}{\partial u}du + \frac{\partial b}{\partial v}dv\right) dv \\ &= \left(\frac{\partial b}{\partial u} - \frac{\partial a}{\partial v}\right) du\,dv. \end{aligned}$$

If $d\phi = \eta$, then we want

$$\frac{\partial b}{\partial u} - \frac{\partial a}{\partial v} = \cos u.$$

If we choose

$$b = \sin u \quad \text{and} \quad a = 0$$

then the requirement is satisfied.

Since the boundary of the northern hemisphere is the equator α defined by $u = 0$, $0 \le v \le 2\pi$, Theorem 6.5 gives

$$\int_H \eta = \int_\alpha \phi.$$

We now apply Theorem 6.2 to ϕ. Since we have $u = 0$ on the equator,

$$\begin{aligned}\int_\alpha \phi &= \int_0^{2\pi} \cos(0)\, dv \\ &= \int_0^{2\pi} dv \\ &= [v]_0^{2\pi} \\ &= 2\pi.\end{aligned}$$

■

The significance of this result is that it is half the surface area of the sphere. (You may know the formula $4\pi r^2$ for the area of a sphere of radius r).

The process of constructing η can be generalized to any patch on any surface. The resulting 2-form is usually known as the *area* 2-form. The example above indicates why: integrating the area 2-form over a region gives the area of that region.

There is a geometric connection with areas that makes the result above intuitively reasonable. Two tangent vectors $\mathbf{v}$ and $\mathbf{w}$ at a point define a parallelogram. The magnitude of their cross product is

$$||\mathbf{v}||\,||\mathbf{w}||\sin\theta,$$

where θ is the angle between them. This is also the (signed) area of the parallelogram defined by the vectors. The effect of the definition

$$\eta(\mathbf{v}, \mathbf{w}) = \mathbf{v} \times \mathbf{w} \cdot U$$

is that $\eta(\mathbf{v}, \mathbf{w})$ gives the area of the parallelogram. Thus η is closely connected with ideas of area.

We shall not pursue integration of forms further. However, the brief discussion above does indicate the links both with previous work and with some that is to follow.

Solutions to the exercises

Solution 1.1

(a) **Cone** The problem here is the point (vertex) of the cone. There is no well-defined tangent plane at the point and so the point cannot be in the image of a patch (let alone a proper patch).

(b) **Closed disc** The points on the rim of the disc cannot meet the requirement of having a neighbourhood contained in the image of a proper patch. They fail precisely because they are right on the edge.

(c) **Folded plane** To understand exactly what goes wrong here, we need to decide what the set is. The condition

$$xy = 0$$

forces $x = 0$ or $y = 0$. Taken together with the condition

$$x \geq 0, \quad y \geq 0$$

this means that we have half the plane $x = 0$ and half the plane $y = 0$. These meet along a sharp edge: the z-axis. Thus M fails the 'smoothness' test.

Solution 1.2

For each mapping we test for being one–one and regular. (Each *is* a mapping because all the coordinate functions are differentiable.)

(a) Suppose that

$$\mathbf{x}(u_1, v_1) = \mathbf{x}(u_2, v_2).$$

Then

$$\begin{aligned} &(u_1, u_1v_1, v_1) = (u_2, u_2v_2, v_2) \\ \Rightarrow\ & u_1 = u_2,\ u_1v_1 = u_2v_2,\ v_1 = v_2 \\ & \text{(comparing coordinates)} \\ \Rightarrow\ & (u_1, v_1) = (u_2, v_2). \end{aligned}$$

Hence $\mathbf{x}$ is one–one.

The Jacobian matrix for $\mathbf{x}_*$ is

$$\begin{pmatrix} 1 & 0 \\ v & u \\ 0 & 1 \end{pmatrix}.$$

Looking at the zeros and ones, we can see that the columns are linearly independent and so the matrix has rank 2. Thus $\mathbf{x}$ is regular.

This completes the proof that $\mathbf{x}$ is a patch.

Note that $\mathbf{x}$ is actually a proper patch since the inverse

$$(p_1, p_2, p_3) \longmapsto (p_1, p_3)$$

has continuous coordinate functions.

(b) First we tackle the one–one property.

$$\begin{aligned} &\mathbf{x}(u_1, v_1) = \mathbf{x}(u_2, v_2) \\ \Rightarrow\ & u_1^2 = u_2^2,\ u_1^3 = u_2^3,\ v_1 = v_2 \\ \Rightarrow\ & u_1^3 = u_2^3 \\ \Rightarrow\ & u_1 = u_2 \\ & (u \longmapsto u^3 \text{ is one–one}) \\ \Rightarrow\ & (u_1, v_1) = (u_2, v_2). \end{aligned}$$

Thus $\mathbf{x}$ is one–one.

The Jacobian is

$$\begin{pmatrix} 2u & 0 \\ 3u^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

The rows are *not* independent in the case $u = 0$ because the matrix reduces to

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $\mathbf{x}$ is not regular everywhere on its domain and is not a patch.

(c) We adopt the same approach as above.

$$\begin{aligned} &\mathbf{x}(u_1, v_1) = \mathbf{x}(u_2, v_2) \\ \Rightarrow\ & (u_1, u_1^2, v_1 + v_1^3) = (u_2, u_2^2, v_2 + v_2^3) \\ \Rightarrow\ & u_1 = u_2,\ u_1^2 = u_2^2,\ v_1 + v_1^3 = v_2 + v_2^3. \end{aligned}$$

From the above we can deduce directly that $u_1 = u_2$. Getting at the vs is slightly more work.

$$\begin{aligned} &v_1 + v_1^3 = v_2 + v_2^3 \\ \Rightarrow\ & v_1 - v_2 + v_1^3 - v_2^3 = 0 \\ \Rightarrow\ & (v_1 - v_2) + (v_1 - v_2)(v_1^2 + v_1v_2 + v_2^2) = 0 \\ \Rightarrow\ & (v_1 - v_2)(1 + v_1^2 + v_1v_2 + v_2^2) = 0. \end{aligned}$$

Now, completing the square shows that

$$\begin{aligned} 1 + v_1^2 + v_1v_2 + v_2^2 &= 1 + \left(v_1 + \frac{v_2}{2}\right)^2 + \frac{3v_2^2}{4} \\ &\geq 1. \end{aligned}$$

Hence we must have

$$v_1 - v_2 = 0$$

$$v_1 = v_2.$$

That completes the proof that $\mathbf{x}$ is one–one.

The Jacobian is

$$\begin{pmatrix} 1 & 0 \\ 2u & 0 \\ 0 & 1 + 3v^2 \end{pmatrix}.$$

The first and last rows show that the matrix has independent columns and so has rank 2. Thus $\mathbf{x}$ is a patch.

Note: It is true, although not obvious, that $\mathbf{x}$ is a proper patch. The inverse of

$$v \longmapsto v + v^3$$

is continuous even though it is difficult to express explicitly.

(d) Because the domain is the whole of $\mathbf{E}^2$, the appearance of $\cos 2\pi u$ prevents $\mathbf{x}$ from being one–one. A counterexample provides a complete proof.

$$\begin{aligned} \mathbf{x}(0, 0) &= (\cos 0, \sin 0, 0) \\ &= (1, 0, 0) \\ &= (\cos 2\pi, \sin 0, 0) \\ &= \mathbf{x}(1, 0). \end{aligned}$$

We would not expect different values of v to be any use in constructing a counterexample because of the appearance of v in the third coordinate function of $\mathbf{x}$.

Since $\mathbf{x}$ is not one–one, it is not a patch. Calculation of the Jacobian would show, however, that $\mathbf{x}$ is regular.

Solution 1.3

We apply the $dg \neq 0$ test in each case.

(a) Here we define

$$g = (x^2 + y^2)^2 + 3z^2$$

and then M becomes

$$M : g = 1$$

and we can apply the test.

$$dg = 2(x^2 + y^2)2x\,dx + 2(x^2 + y^2)2y\,dy + 6z\,dz.$$

The only way we can have $dg = 0$ is when

$$x = y = z = 0.$$

However, $x = y = z = 0$ do not satisfy $g = 1$ and so dg cannot be zero on M. Hence M is a surface.

(b) We consider

$$g = z(z - 2) + xy.$$

We have

$$dg = y\,dx + x\,dy + 2(z - 1)\,dz.$$

We have $dg = 0$ when

$$x = y = 0, z = 1.$$

Now, M will be a surface provided that these values do *not* satisfy $g = c$. We have

$$g(0, 0, 1) = -1.$$

So, M is a surface for all values of c *except* $c = -1$.

Solution 1.4

We first show that $\mathbf{x}$ is one–one.

$$\mathbf{x}(u_1, v_1) = \mathbf{x}(u_2, v_2)$$
$$\Rightarrow u_1^2 = u_2^2,\ u_1 v_1 = u_2 v_2,\ v_1^2 = v_2^2.$$

Now, because of the restriction of the domain to *positive* values of u and v, we can deduce $u_1 = u_2$ and $v_1 = v_2$ from the equality of their squares. (Note how the property of being one–one depends on the domain as well as the formulas.)

We also have that $\mathbf{x}$ is regular because the Jacobian is

$$\begin{pmatrix} 2u & 0 \\ v & u \\ 0 & 2v \end{pmatrix}.$$

The only values that would give this a rank of less than 2 are $u = v = 0$ which are excluded from the domain.

The formula for the inverse is obtained from

$$(u^2, uv, v^2) = (p_1, p_2, p_3)$$

to give

$$\mathbf{x}^{-1} : (p_1, p_2, p_3) \longmapsto (\sqrt{p_1}, \sqrt{p_3}).$$

Or

$$\mathbf{x}^{-1} = (\sqrt{x}, \sqrt{z}).$$

From this, $\mathbf{x}^{-1}$ is continuous and $\mathbf{x}$ is a proper patch.

Solution 1.5

(a) If we rewrite the definition of M as

$$M : \frac{x^2 - y^2}{4} - z = 0,$$

We can see that this is

$$M : g = c,$$

where

$$g = \frac{x^2 - y^2}{4} - z \quad \text{and} \quad c = 0.$$

Now

$$dg = \tfrac{1}{2}x\,dx - \tfrac{1}{2}y\,dy - dz$$

which cannot be zero because of the third term. Hence M is a surface.

(b) We must show that $\mathbf{x}(u, v)$ satisfies the equation for M. Since a point $\mathbf{p}$ lies on M if, and only if, it satisfies

$$\left(\frac{x^2 - y^2)}{4} - z\right)(\mathbf{p}) = 0,$$

we evaluate that function on $\mathbf{x}(u, v)$. We have

$$\begin{aligned}
&\left(\frac{x^2 - y^2}{4} - z\right)(\mathbf{x}(u, v)) \\
&= \left(\frac{x^2 - y^2}{4} - z\right)(u + v, u - v, uv) \\
&= \tfrac{1}{4}\left((u + v)^2 - (u - v)^2\right) - uv \\
&= \tfrac{1}{4}(u^2 + 2uv + v^2 - u^2 + 2uv - v^2) - uv \\
&= \tfrac{1}{4}4uv - uv \\
&= 0.
\end{aligned}$$

Thus $\mathbf{x}(u, v)$ lies in M for all u and v.

(c) To show that $\mathbf{x}$ is onto M we must show that

$$\mathbf{x}(u, v) = (p_1, p_2, p_3)$$

can be solved for u and v for any point

$$\mathbf{p} = (p_1, p_2, p_3)$$

on M.

We require

$$\begin{aligned}
u + v &= p_1 \\
u - v &= p_2 \\
uv &= p_3.
\end{aligned}$$

The first two equations can be solved to give

$$\begin{aligned}
u &= \tfrac{1}{2}(p_1 + p_2) \\
v &= \tfrac{1}{2}(p_1 - p_2).
\end{aligned}$$

It remains to show that these values also satisfy the third equation.

$$\begin{aligned}
uv &= \tfrac{1}{4}(p_1 + p_2)(p_1 - p_2) \\
&= \tfrac{1}{4}(p_1^2 - p_2^2) \\
&= p_3.
\end{aligned}$$

The last step follows because $\mathbf{p}$ lies on M and so satisfies the equation for M.

(d) The fact that we were able to solve the equations above *uniquely* shows that $\mathbf{x}$ is one–one. The inverse

$$(p_1, p_2, p_2) \longmapsto \left(\tfrac{1}{2}(p_1 + p_2), \tfrac{1}{2}(p_1 - p_2)\right)$$

is continuous.

It remains to show that $\mathbf{x}$ is regular. The Jacobian is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ v & u \end{pmatrix}.$$

The ones and minus ones show that the columns are independent and the matrix has rank 2.

This completes the proof that $\mathbf{x}$ is a proper patch.

Solution 2.1

(a) The technique from Example 2.5 of *O'Neill* applies directly. We have a parametrization of C as

$$\alpha(t) = (t, \cosh t, 0)$$

so

$$g(u) = u, \quad h(u) = \cosh u,$$

leading to the parametrization

$$\mathbf{x}(u, v) = (u, \cosh u \cos v, \cosh u \sin v).$$

(b) We must first parametrize C. Now, C is a circle, centre $(0, 0, 2)$, radius 1 in the yz-plane. This suggests

$$\alpha(t) = (0, \cos t, 2 + \sin t).$$

We are rotating about the y-axis, so the function corresponding to g in Example 2.5 is the y-coordinate function. This leads to the parametrization

$$\mathbf{x}(u, v) = ((2 + \sin u) \cos v, \cos u, (2 + \sin u) \sin v).$$

Note: In any of these parametrizations the roles of $\cos v$ and $\sin v$ may be interchanged if you wish.

(c) The way in which C is defined suggests the following parametrization of C.

$$\alpha(t) = (t, 0, t^2).$$

Since we are rotating about the z-axis, the g function is the third coordinate function of α. This gives

$$\mathbf{x}(u, v) = (u \cos v, u \sin v, u^2).$$

Solution 2.2

We know that the norm of the cross product is given by

$$\|\mathbf{x}_u \times \mathbf{x}_v\|^2 = \|\mathbf{x}_u\|^2 \|\mathbf{x}_v\|^2 \sin^2 \theta,$$

where θ is the angle between the partial velocities.

However, we also know that

$$\begin{aligned} E &= \mathbf{x}_u \cdot \mathbf{x}_u \\ &= \|\mathbf{x}_u\|^2; \\ F &= \mathbf{x}_u \cdot \mathbf{x}_v \\ &= \|\mathbf{x}_u\| \, \|\mathbf{x}_v\| \cos \theta; \\ G &= \mathbf{x}_v \cdot \mathbf{x}_v \\ &= \|\mathbf{x}_v\|^2. \end{aligned}$$

Hence

$$\begin{aligned} \|\mathbf{x}_u \times \mathbf{x}_v\|^2 &= \|\mathbf{x}_u\|^2 \|\mathbf{x}_v\|^2 \sin^2 \theta \\ &= \|\mathbf{x}_u\|^2 \|\mathbf{x}_v\|^2 (1 - \cos^2 \theta) \\ &= EG - F^2. \end{aligned}$$

Since $\mathbf{x}$ is regular if, and only if,

$$\mathbf{x}_u \times \mathbf{x}_v \neq 0,$$

it follows that $\mathbf{x}$ is regular if, and only if,

$$EG - F^2 \neq 0.$$

Solution 2.3

We start by calculating the partial velocities.

$$\begin{aligned} \mathbf{x}_u(u, v) &= 0 + v\delta'(u) \\ &= v\delta'(u); \\ \mathbf{x}_v(u, v) &= 0 + \delta(u) \\ &= \delta(u). \end{aligned}$$

The mapping $\mathbf{x}$ will be regular if, and only if, the cross product is non-zero. We have

$$\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) = v\delta'(u) \times \delta(u).$$

Thus x is regular wherever both v and $\delta' \times \delta$ are non-zero, as claimed.

Solution 2.4

Using the same approach as in the previous solution we have:

$$\begin{aligned} \mathbf{x}_u(u, v) &= \beta'(u), \\ \mathbf{x}_v(u, v) &= \mathbf{q}, \\ \mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) &= \beta'(u) \times \mathbf{q}. \end{aligned}$$

The result follows.

Solution 2.5

(a) We start by tackling regularity and then deal with the one–one property.

$$\begin{aligned} \mathbf{x}_u(u, v) &= (\cos v, \sin v, 0), \\ \mathbf{x}_v(u, v) &= (-u \sin v, u \cos v, b), \\ \mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) &= (b \sin v, -b \cos v, u). \end{aligned}$$

We are given $b \neq 0$ and $\cos v$ and $\sin v$ cannot be zero simultaneously. Thus

$$\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) \neq 0$$

and $\mathbf{x}$ is regular everywhere.

Now we prove that $\mathbf{x}$ is one–one. Suppose that

$$(u_1 \cos v_1, u_1 \sin v_1, bv_1) = (u_2 \cos v_2, u_2 \sin v_2, bv_2).$$

Since $b \neq 0$, comparing the last coordinates gives

$$v_1 = v_2.$$

Since at least one of $\cos v_1$ or $\sin v_1$ is non-zero, we can deduce that

$$u_1 \cos v_1 = u_2 \cos v_2 = u_2 \cos v_1$$

or

$$u_1 \sin v_1 = u_2 \sin v_2 = u_2 \sin v_1.$$

In either case,

$$u_1 = u_2.$$

Thus $\mathbf{x}$ is one–one and regular, hence a patch.

(b) Fixing $v = v_0$ gives

$$\begin{aligned} \mathbf{x}(u, v_0) &= (u \cos v_0, u \sin v_0, bv_0) \\ &= (0, 0, bv_0) + u(\cos v_0, \sin v_0, 0), \end{aligned}$$

which indicates that the u-parameter curves are straight lines.

Fixing $u = u_0$ gives

$$\mathbf{x}(u_0, v) = (u_0 \cos v, u_0 \sin v, bv),$$

which defines a (circular) helix. (This gives some indication of where the name *helicoid* comes from.)

(c) Provided that we avoid places where $\cos v = 0$ or $u = 0$, we can write

$$\frac{u \sin v}{u \cos v} = \tan v.$$

This suggests an implicit equation of the form

$$\frac{y}{x} = \tan \frac{z}{b}.$$

We can restore the excluded values by rewriting this as

$$y = x \tan \frac{z}{b}.$$

Thus

$$H : g = c,$$

where

$$g = y - x \tan \frac{z}{b}, \quad c = 0.$$

Solution 2.6

We can actually use a single proof to show that all three are surfaces. Each is given in the form

$$g = c$$

and we have

$$dg = \frac{2x}{a^2}\,dx + \frac{2y}{b^2}\,dy \pm \frac{2z}{c^2}\,dz.$$

In each case, $dg = 0$ if, and only if

$$x = y = z = 0.$$

In none of the cases does the origin satisfy

$$g = c.$$

It follows that all three are surfaces in $\mathbf{E}^3$.

Now we tackle the parametrizations and images for each surface in turn. We assume that, in all cases,

$$a, b, c > 0.$$

To show that $\mathbf{x}$ is a parametrization, we must show that $\mathbf{x}$ is regular and that each point of the image lies in the surface. To shorten some of the long expressions, we shall omit the parameters (u, v).

(a) We apply the cross product test for regularity.

$$\begin{aligned}
\mathbf{x}_u &= (-a\sin u\cos v, -b\sin u\sin v, c\cos u),\\
\mathbf{x}_v &= (-a\cos u\sin v, b\cos u\cos v, 0),\\
\mathbf{x}_u \times \mathbf{x}_v &= (-bc\cos^2 u\cos v, -ac\cos^2 u\sin v, -ab\sin u\cos u)
\end{aligned}$$

For the values of u given,

$$\cos u \neq 0$$

and $\cos v$ and $\sin v$ cannot be zero simultaneously, so the cross product is non-zero. Hence $\mathbf{x}$ is regular on D.

Also, $\mathbf{x}(D)$ lies in M because

$$\begin{aligned}
\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)(\mathbf{x}) &= \cos^2 u\cos^2 v + \cos^2 u\sin^2 v\\
&\quad + \sin^2 u\\
&= \cos^2 u(\cos^2 v + \sin^2 v)\\
&\quad + \sin^2 u\\
&= \cos^2 u + \sin^2 u\\
&= 1.
\end{aligned}$$

The restriction

$$-\frac{\pi}{2} < u < \frac{\pi}{2}$$

means that $\sin u$ lies in the range

$$-1 < \sin u < 1$$

and $\cos u$ in the range

$$0 < \cos u \leq 1.$$

Since v is unrestricted, we have

$$\begin{aligned}
&-a \leq a\cos u\cos v \leq a,\\
&-b \leq b\cos u\sin v \leq b\\
&-c < c\sin u < c.
\end{aligned}$$

Thus the image $\mathbf{x}(D)$ is all of M *except* the two points $(0, 0, \pm c)$.

(b) Here

$$\begin{aligned}
\mathbf{x}_u &= (a\sinh u\cos v, b\sinh u\sin v, c\cosh u),\\
\mathbf{x}_v &= (-a\cosh u\sin v, b\cosh u\cos v, 0),\\
\mathbf{x}_u \times \mathbf{x}_v &= (-bc\cosh^2 u\cos v, -ac\cosh^2 u\sin v,\\
&\qquad ab\cosh u\sinh u)\\
&= \cosh u(-bc\cosh u\cos v, -ac\cosh u\sin v,\\
&\qquad ab\sinh u).
\end{aligned}$$

Now, $\cosh u$ is never zero and $\cos v$ and $\sin v$ cannot be simultaneously zero so the first two components cannot both be zero and $\mathbf{x}$ is regular.

Now

$$\begin{aligned}
\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)(\mathbf{x}) &= \cosh^2 u\cos^2 v + \cosh^2 \sin^2 v - \sinh^2\\
&= \cosh^2 u(\cos^2 v + \sin^2 v) - \sinh^2 u\\
&= \cosh^2 u - \sinh^2 u\\
&= 1.
\end{aligned}$$

Hence $\mathbf{x}(\overset{D}{\cancel{\mathbf{R}}}) \subseteq M$ and $\mathbf{x}$ is a parametrization of M.

The image $\mathbf{x}(D)$ is the whole of M.

(c) Using the same approach as above:

$$\begin{aligned}
\mathbf{x}_u &= (a\cosh u\cos v, b\cosh u\sin v, c\sinh u),\\
\mathbf{x}_v &= (-a\sinh u\sin v, b\sinh u\cos v, 0),\\
\mathbf{x}_u \times \mathbf{x}_v &= (-bc\sinh^2 u\cos v, -ac\sinh^2 u\sin v, ab\sinh u\cosh u\\
&= \sinh u(-bc\sinh u\cos v, -ac\sinh u\sin v, ab\cosh u).
\end{aligned}$$

The restriction $u \neq 0$ ensures that $\sinh u$ is non-zero, $\cosh u$ is never zero and $\cos v$ and $\sin v$ cannot be simultaneously zero. Hence $\mathbf{x}$ is regular.

We have

$$\begin{aligned}
\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)(\mathbf{x}) &= \sinh^2 u\cos^2 v + \sinh^2 u\sin^2 v - \cosh^2\\
&= \sinh^2 u - \cosh^2 u\\
&= -1.
\end{aligned}$$

So $\mathbf{x}(D) \subseteq M$ and $\mathbf{x}$ is a parametrization of M.

The restriction $u \neq 0$ means that

$$c\cosh u > c$$

and so

$$\frac{z^2}{c^2} > 1.$$

This excludes the two points

$$(0, 0, \pm c),$$

both of which satisfy the equation for M. Thus the image is all of M except these two points.

Solution 2.7

(a) We can use the $dg \neq 0$ test if we rewrite the definition of M as

$$M : g = 0,$$

where

$$g = \frac{x^2}{a^2} + \frac{y^2}{b^2} - z.$$

We have

$$dg = \frac{2x}{a^2}\,dx + \frac{2y}{b^2}\,dy - dz \neq 0.$$

Hence M is a surface.

Next, we show that $\mathbf{x}$ is a parametrization.

$$\begin{aligned}
\mathbf{x}_u &= (a\cos v, b\sin v, 2u),\\
\mathbf{x}_v &= (-au\sin v, bu\cos v, 0),\\
\mathbf{x}_u \times \mathbf{x}_v &= (-2bu^2\cos v, -2au^2\sin v, abu).
\end{aligned}$$

Since $u > 0$ and $\cos v$ and $\sin v$ cannot both be zero, $\mathbf{x}$ is regular.

Since

$$\begin{aligned}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)(\mathbf{x}) &= u^2\cos^2 v + u^2\sin^2 v\\ &= u^2\\ &= z(\mathbf{x}(u,v)),\end{aligned}$$

the image set of $\mathbf{x}$ is a subset of M. Hence $\mathbf{x}$ is a parametrization of M.

The restriction $u > 0$ excludes the point $(0,0,0)$ which lies in M.

(b) Points on the parameter curve $u = u_0$ have the form

$$(au_0\cos v, bu_0\sin v, u_0^2)$$

and satisfy

$$\frac{x^2}{a^2u_0^2}+\frac{y^2}{b^2u_0^2}=1, \quad z = u_0^2.$$

These curves are, therefore, ellipses.

Points on the parameter curve $v = v_0$ have the form

$$\begin{aligned}\mathbf{x}(u,v_0) &= (au\cos v_0, bu\sin v_0, u^2)\\ &= u(a\cos v_0, b\sin v_0, 0) + u^2(0,0,1).\end{aligned}$$

The parameter curve, therefore, lies in the plane defined by the two vectors

$$(a\cos v_0, b\sin v_0, 0) \quad \text{and} \quad (0,0,1).$$

If we introduce a 'w-axis' in the direction of

$$(a\cos v_0, b\sin v_0, 0),$$

the equation of the parameter curve becomes

$$z = w^2.$$

This shows that the parameter curve is a parabola.

The nature of the parameter curves indicates why the surface is called an elliptic paraboloid.

Solution 2.8

(a) We must show that $\mathbf{x}$ is a proper patch and onto M.

We have

$$\begin{aligned}\mathbf{x}_u &= (a,b,4v),\\ \mathbf{x}_v &= (a,-b,4u),\\ \mathbf{x}_u\times\mathbf{x}_v &= (4b(u+v), 4a(v-u), -2ab).\end{aligned}$$

Since, for the definition of M to make sense, $a, b \neq 0$ the cross product is non-zero and $\mathbf{x}$ is regular.

Since

$$\begin{aligned}\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)(\mathbf{x}) &= (u+v)^2-(u-v)^2\\ &= u^2+2uv+v^2-u^2+2uv-v^2\\ &= 4uv\\ &= z(\mathbf{x}),\end{aligned}$$

the image set of $\mathbf{x}$ is a subset of M and $\mathbf{x}$ is a parametrization.

Finally, suppose that

$$\mathbf{p} = (p_1, p_2, p_3)$$

belongs to M. Then we can solve

$$\begin{aligned}p_1 &= a(u+v)\\ p_2 &= b(u-v)\end{aligned}$$

to obtain

$$\begin{aligned}u &= \frac{1}{2}\left(\frac{p_1}{a}+\frac{p_2}{b}\right)\\ v &= \frac{1}{2}\left(\frac{p_1}{a}-\frac{p_2}{b}\right).\end{aligned}$$

We have expressed $\mathbf{p}$ in the form

$$\mathbf{p} = \mathbf{x}(u,v)$$

for suitable u and v and so $\mathbf{x}$ is onto M.

(b) The two ruled forms are

$$\begin{aligned}\mathbf{x}(u,v) &= (au, bu, 0) + v(a,-b,4u)\\ \mathbf{x}(u,v) &= (av, -bv, 0) + u(a,b,4v).\end{aligned}$$

(c) Writing the parameter curves in the forms

$$\begin{aligned}\mathbf{x}(u_0,v) &= (au_0, bu_0, 0) + v(a,-b,4u_0),\\ \mathbf{x}(u,v_0) &= (av_0, -bv_0, 0) + u(a,b,4v_0),\end{aligned}$$

shows that they are straight lines.

Solution 3.1

We calculate the expressions $f(\mathbf{x})$ in each case.

(a) Here we have $f = x^2+y^2$, so

$$\begin{aligned}f(\mathbf{x}(u,v)) &= r^2\cos^2 v\cos^2 u + r^2\cos^2 v\sin^2 u\\ &= r^2\cos^2 v(\cos^2 u+\sin^2 u)\\ &= r^2\cos^2 v.\end{aligned}$$

(b) This time

$$\begin{aligned}f(\mathbf{x}(u,v)) &= (r\cos v\cos u - r\cos v\sin u)^2 + r^2\sin^2 v\\ &= r^2\cos^2 v(\cos^2 u+\sin^2 u - 2\cos u\sin u)\\ &\quad + r^2\sin^2 v\\ &= r^2\cos^2 v(1-\sin 2u) + r^2\sin^2 v\\ &= r^2 - r^2\cos^2 v\sin 2u\\ &= r^2(1-\cos^2 v\sin 2u).\end{aligned}$$

Solution 3.2

(a) We calculate $\mathbf{x}(\alpha(t))$.

$$\mathbf{x}(\alpha(t)) = ((R+r\cos t)\cos t, (R+r\cos t)\sin t, r\sin t).$$

Thus

$$\begin{aligned}\alpha_1(t) &= (R+r\cos t)\cos t,\\ \alpha_2(t) &= (R+r\cos t)\sin t,\\ \alpha_3(t) &= r\sin t.\end{aligned}$$

(b) Since the coordinate functions involve $\sin t$ and $\cos t$, they will repeat after 2π, that is,

$$\alpha(t+2\pi) = \alpha(t),$$

for all values of t. The only question that arises is whether they can repeat earlier.

So, suppose that

$$\alpha(t) = \alpha(t'),$$

for some values of t and t'. Now, looking at the third coordinate function of α, we can deduce that

$$\sin t = \sin t'.$$

If $\sin t = \sin t'$ is not zero, then the second coordinate function shows that

$$R + r\cos t = R + r\cos t'$$

and hence

$$\cos t = \cos t'.$$

On the other hand, if

$$\sin t = \sin t' = 0,$$

then

$$\cos t = \pm 1, \ \cos t' = \pm 1.$$

Since $r < R$, the expressions

$$R + r\cos t, \ R + r\cos t'$$

are both positive and so the equation

$$R + r\cos t = R + r\cos t'$$

forces $\cos t$ and $\cos t'$ to have the same sign. Thus

$$\cos t = \cos t' = \pm 1.$$

Summing up,

$$\sin t = \sin t'$$

and, regardless of whether or not this common value is zero,

$$\cos t = \cos t'.$$

These equations are satisfied only if

$$t' = t + 2n\pi,$$

for some integer n.

Hence α is periodic with period 2π.

Solution 3.3

We have to apply the basic test: that $\mathbf{v}$ is tangent to M if, and only if, we can write $\mathbf{v}$ as a linear combination of the partial velocities at its point of application.

We begin by defining a parametrization; the appropriate one here is the Monge patch.

$$\mathbf{x}(u, v) = (u, v, f(u, v)).$$

(a) First we calculate the partial velocities.

$$\mathbf{x}_u(u, v) = \left(1, 0, \frac{\partial f}{\partial u}(u, v)\right),$$
$$\mathbf{x}_v(u, v) = \left(0, 1, \frac{\partial f}{\partial v}(u, v)\right).$$

Now, suppose that

$$\mathbf{v}_\mathbf{p} = (v_1, v_2, v_3)_{(p_1, p_2, p_3)}.$$

Then

$$\mathbf{p} = \mathbf{x}(p_1, p_2).$$

We also note that, for the Monge patch, partial differentiation with respect to u and v is the same as with respect to x and y.

Thus, $\mathbf{v}$ is tangent to M if, and only if, $\mathbf{v}$ can be written as a linear combination of $\mathbf{x}_u(p_1, p_2)$ and $\mathbf{x}_v(p_1, p_2)$.

Inspecting the expressions for the partial velocities above, this condition reduces to

$$(v_1, v_2, v_3) = v_1\mathbf{x}_u(p_1, p_2) + v_2\mathbf{x}_v(p_1, p_2).$$

Comparing the third coordinates and using the observation about partial derivatives, we arrive at

$$v_3 = \frac{\partial f}{\partial x}(p_1, p_2)v_1 + \frac{\partial f}{\partial y}(p_1, p_2)v_2.$$

(b) We use the observation made in the last section that

$$\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)$$

is normal to the surface at $\mathbf{x}(u, v)$.

Now $\mathbf{v}$ is tangent to M at $\mathbf{x}(u, v)$ if, and only if, it is perpendicular to the normal at $\mathbf{x}(u, v)$. This is true if, and only if,

$$\mathbf{v} \cdot \mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) = 0.$$

Solution 3.4

(a) The easiest approach is to use the chain rule because it automatically expresses α' in terms of the partial velocities.

$$\begin{aligned}\alpha'(t) &= \mathbf{x}_u(\sqrt{2}t, e^t)\frac{d\sqrt{2}t}{dt} + \mathbf{x}_v(\sqrt{2}t, e^t)\frac{de^t}{dt}\\ &= \sqrt{2}\mathbf{x}_u(\sqrt{2}t, e^t) + e^t\mathbf{x}_v(\sqrt{2}t, e^t).\end{aligned}$$

Note: We have not needed to find explicit expressions for the partial velocities yet.

(b) If we use the result just found, we have

$$\alpha' \cdot \mathbf{x}_u = \sqrt{2}\mathbf{x}_u \cdot \mathbf{x}_u + e^t\mathbf{x}_v \cdot \mathbf{x}_u,$$
$$\alpha' \cdot \mathbf{x}_v = \sqrt{2}\mathbf{x}_u \cdot \mathbf{x}_v + e^t\mathbf{x}_v \cdot \mathbf{x}_v.$$

We also have

$$\begin{aligned}\mathbf{x}_u(u, v) &= v(-\sin u, \cos u, 0),\\ \mathbf{x}_v(u, v) &= (\cos u, \sin u, 1),\\ \mathbf{x}_u(u, v) \cdot \mathbf{x}_u(u, v) &= v^2,\\ \mathbf{x}_u(u, v) \cdot \mathbf{x}_v(u, v) &= 0,\\ \mathbf{x}_v(u, v) \cdot \mathbf{x}_v(u, v) &= 2.\\ \|\mathbf{x}_u(u, v)\| &= |v|,\\ \|\mathbf{x}_v(u, v)\| &= \sqrt{2}.\end{aligned}$$

Putting these results together, we obtain

$$\begin{aligned}\alpha' \cdot \mathbf{x}_u/\|\mathbf{x}_u\| &= \sqrt{2}(e^t)^2/e^t\\ &= \sqrt{2}e^t,\\ \alpha' \cdot \mathbf{x}_v/\|x_v\| &= e^t \times 2/\sqrt{2}\\ &= \sqrt{2}e^t.\end{aligned}$$

Thus α' makes equal angles with the unit vectors

$$\frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} \text{ and } \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

and the result follows.

Solution 3.5

In order to apply the result

$$\mathbf{x}_u[f] = \frac{\partial f(\mathbf{x})}{\partial u}$$

we need to find $f(\mathbf{x}(u, v))$.

Now

$$\begin{aligned}f(\mathbf{x}(u, v)) &= f(u, v, uv)\\ &= u^2 + v^2 - uv.\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{x}_u[f] &= \frac{\partial f(\mathbf{x}(u, v))}{\partial u}\\ &= \frac{\partial}{\partial u}(u^2 + v^2 - uv)\\ &= 2u - v.\end{aligned}$$

Similarly

$$\begin{aligned}\mathbf{x}_v[f] &= \frac{\partial f(\mathbf{x}(u, v))}{\partial v}\\ &= \frac{\partial}{\partial v}(u^2 + v^2 - uv)\\ &= 2v - u.\end{aligned}$$

Solution 3.6

First we calculate $\mathbf{y}(u, v)$ in terms of u and v.

$$\begin{aligned}\mathbf{y}(u, v) &= \mathbf{x}(u^2 + v^2, u^2 - v^2)\\ &= (u^2 + v^2, u^2 - v^2, (u^2 + v^2)(u^2 - v^2))\\ &= (u^2 + v^2, u^2 - v^2, u^4 - v^4).\end{aligned}$$

The four partial velocities are

$$\begin{aligned}\mathbf{x}_u(u, v) &= (1, 0, v),\\ \mathbf{x}_v(u, v) &= (0, 1, u),\\ \mathbf{y}_u(u, v) &= (2u, 2u, 4u^3),\\ \mathbf{y}_v(u, v) &= (2v, -2v, -4v^3).\end{aligned}$$

In the notation used above for deriving the chain rule, we have

$$\overline{u}(u,v) = u^2 + v^2,$$
$$\overline{v}(u,v) = u^2 - v^2.$$

Hence

$$\frac{\partial \overline{u}}{\partial u} = 2u,$$
$$\frac{\partial \overline{u}}{\partial v} = 2v,$$
$$\frac{\partial \overline{v}}{\partial u} = 2u,$$
$$\frac{\partial \overline{v}}{\partial v} = -2v.$$

Using the chain rule

$$\mathbf{y}_u(u,v) = \mathbf{x}_u(\tilde{u},\tilde{v})\frac{\partial \tilde{u}}{\partial u} + \mathbf{x}_v(\tilde{u},\tilde{v})\frac{\partial \tilde{v}}{\partial u}$$

to calculate $\mathbf{y}_u$ gives

$$\begin{aligned}\mathbf{y}_u(u,v) &= \mathbf{x}_u(u^2+v^2, u^2-v^2) \times 2u \\ &\quad + \mathbf{x}_v(u^2+v^2, u^2 - v_2) \times 2u \\ &= 2u(1,0,u^2-v^2) + 2u(0,1,u^2+v^2) \\ &= (2u, 2u, 4u^3).\end{aligned}$$

A similar calculation gives

$$\begin{aligned}\mathbf{y}_v(u,v) &= \mathbf{x}_u(u^2+v^2, u^2-v^2) \times 2v \\ &\quad + \mathbf{x}_v(u^2+v^2, u^2 - v_2) \times (-2v) \\ &= 2v(1,0,u^2-v^2) - 2v(0,1,u^2+v^2) \\ &= (2v, -2v, -4v^3).\end{aligned}$$

These correspond to the results obtained by direct calculation.

Solution 4.1

We can show equality of 2-forms by evaluating them on a pair of linearly independent vectors at each point of the surface. For each patch $\mathbf{x}$ in the surface, we have

$$\begin{aligned}(\phi \wedge \psi)(\mathbf{x}_u, \mathbf{x}_v) &= \phi(\mathbf{x}_u)\psi(\mathbf{x}_v) - \phi(\mathbf{x}_v)\psi(\mathbf{x}_u) \\ &= -(\psi(\mathbf{x}_u)\phi(\mathbf{x}_v) - \psi(\mathbf{x}_v)\phi(\mathbf{x}_u)) \\ &= -(\psi \wedge \phi)(\mathbf{x}_u, \mathbf{x}_v).\end{aligned}$$

The result follows.

Note: We have chosen to use the partial velocities, you could have chosen any pair $(\mathbf{v}, \mathbf{w})$ of linearly independent tangent vectors at each point. The details of the argument are exactly the same.

Since we have

$$\phi \wedge \phi = -\phi \wedge \phi,$$

it follows that $\phi \wedge \phi = 0$.

Solution 4.2

We shall need the partial velocities, so we calculate them first.

$$\mathbf{x}_u = (1,0,v),$$
$$\mathbf{x}_v = (0,1,u).$$

(a) By inspection of the first two coordinates we have

$$\mathbf{p} = (-1,3,-1) = \mathbf{x}(-1,3).$$

Hence $\mathbf{p} \in M$.

We have

$$\mathbf{x}_u(-1,3) = (1,0,3),$$
$$\mathbf{x}_v(-1,3) = (0,1,-1).$$

Since

$$\mathbf{v}_\mathbf{p} = \mathbf{x}_u(-1,3) + 2\mathbf{x}_v(-1,3),$$

it follows that $\mathbf{v}_\mathbf{p}$ is tangent to M.

(b) Applying the definition of ϕ:

$$\begin{aligned}\phi(\mathbf{v}_\mathbf{p}) &= \mathbf{v} \cdot \mathbf{x}_u(-1,3) \\ &= (1,2,1) \cdot (1,0,3) \\ &= 4.\end{aligned}$$

$$\begin{aligned}\psi(\mathbf{v}_\mathbf{p}) &= \mathbf{v} \cdot \mathbf{x}_v(-1,3) \\ &= (1,2,1) \cdot (0,1,-1) \\ &= 1.\end{aligned}$$

(c) Again, using the definition,

$$\begin{aligned}\phi(\mathbf{x}_u) &= \mathbf{x}_u(u,v) \cdot \mathbf{x}_u(u,v) \\ &= (1,0,v) \cdot (1,0,v) \\ &= 1 + v^2.\end{aligned}$$

$$\begin{aligned}\phi(\mathbf{x}_v) &= \mathbf{x}_v(u,v) \cdot \mathbf{x}_u(u,v) \\ &= (0,1,u) \cdot (1,0,v) \\ &= uv.\end{aligned}$$

$$\begin{aligned}\psi(\mathbf{x}_u) &= \mathbf{x}_u(u,v) \cdot \mathbf{x}_v(u,v) \\ &= (1,0,v) \cdot (0,1,u) \\ &= uv.\end{aligned}$$

$$\begin{aligned}\psi(\mathbf{x}_v) &= \mathbf{x}_v(u,v) \cdot \mathbf{x}_v(u,v) \\ &= (0,1,u) \cdot (0,1,u) \\ &= 1 + u^2.\end{aligned}$$

(d) We have

$$\begin{aligned}\phi(\mathbf{x}_u) &= (a\,du + b\,dv)(\mathbf{x}_u) \\ &= a\,du(\mathbf{x}_u) + b\,dv(\mathbf{x}_u) \\ &= a \times 1 + b \times 0 \\ &= a.\end{aligned}$$

$$\begin{aligned}\phi(\mathbf{x}_v) &= (a\,du + b\,dv)(\mathbf{x}_v) \\ &= a\,du(\mathbf{x}_v) + b\,dv(\mathbf{x}_v) \\ &= a \times 0 + b \times 1 \\ &= b.\end{aligned}$$

Comparing these to the results from the previous part gives

$$a = (1 + v^2),\ b = uv.$$

Thus

$$\phi = (1 + v^2)du + uv\,dv.$$

(e) If we assume that

$$\psi = a\,du + b\,dv,$$

then exactly the same method shows that

$$a = uv,\ b = (1 + u^2).$$

Hence

$$\psi = uv\,du + (1 + u^2)dv.$$

Solution 4.3

Applying the results obtained above:

$$\begin{aligned}\phi \wedge \psi &= ((1+v^2)du + uv\,dv) \wedge (uv\,du + (1+u^2)dv) \\ &= (1+v^2)(1+u^2)du\,dv + (uv)(uv)dv\,du \\ &\quad \text{(since } du\,du = dv\,dv = 0\text{)} \\ &= (1+v^2)(1+u^2)du\,dv - (uv)(uv)du\,dv \\ &\quad \text{(using alternation)} \\ &= (1 + u^2 + v^2)du\,dv.\end{aligned}$$

Since

$$(du\,dv)(\mathbf{x}_u, \mathbf{x}_v) = 1,$$

we have

$$\begin{aligned}(\phi \wedge \psi)(\mathbf{x}_u, \mathbf{x}_v) &= (1 + u^2 + v^2)(du\,dv)(\mathbf{x}_u, \mathbf{x}_v)\\ &= 1 + u^2 + v^2.\end{aligned}$$

Calculating directly from Definition 4.3 gives

$$\begin{aligned}(\phi \wedge \psi)(\mathbf{x}_u, \mathbf{x}_v) &= \phi(\mathbf{x}_u)\psi(\mathbf{x}_v) - \phi(\mathbf{x}_v)\psi(\mathbf{x}_u)\\ &= (1 + v^2)(1 + u^2) - (uv)(uv)\\ &= 1 + u^2 + v^2,\end{aligned}$$

as before.

Solution 5.1

From the earlier work on this example, we know that

$$\begin{aligned}F_*(\mathbf{x}_u(u, v)) &= \mathbf{y}_u(u, \sin v),\\ F_*(\mathbf{x}_v(u, v)) &= \cos u\, \mathbf{y}_v(u, \sin v).\end{aligned}$$

We shall also need the partial velocities of $\mathbf{y}$.

$$\begin{aligned}\mathbf{y}_u(u, v) &= (-\sin u, \cos u, 0),\\ \mathbf{y}_v(u, v) &= (0, 0, 1).\end{aligned}$$

(a) From the above, we have

$$\begin{aligned}\phi_1(\mathbf{y}_u(u, v)) &= \mathbf{y}_u(u, v) \cdot \mathbf{y}_u(u, v)\\ &= 1;\\ \phi_1(\mathbf{y}_v(u, v)) &= \mathbf{y}_v(u, v) \cdot \mathbf{y}_u(u, v)\\ &= 0;\\ \phi_2(\mathbf{y}_u(u, v)) &= \mathbf{y}_u(u, v) \cdot \mathbf{y}_v(u, v)\\ &= 0;\\ \phi_2(\mathbf{y}_v(u, v)) &= \mathbf{y}_v(u, v) \cdot \mathbf{y}_v(u, v)\\ &= 1.\end{aligned}$$

It follows that

$$\begin{aligned}(F^*\phi_1)(\mathbf{x}_u(u, v)) &= \phi_1(F_*(\mathbf{x}_u(u, v)))\\ &= \phi_1(\mathbf{y}_u(u, \sin v))\\ &= 1.\\ (F^*\phi_1)(\mathbf{x}_v(u, v)) &= \phi_1(F_*(\mathbf{x}_v(u, v)))\\ &= \phi_1(\cos v\, \mathbf{y}_v(u, \sin v))\\ &= \cos v\, \phi_1(\mathbf{y}_u(u, \sin v))\\ &= 0.\\ (F^*\phi_2)(\mathbf{x}_u(u, v)) &= \phi_2(F_*(\mathbf{x}_u(u, v)))\\ &= \phi_2(\mathbf{y}_u(u, \sin v))\\ &= 0.\\ (F^*\phi_2)(\mathbf{x}_v(u, v)) &= \phi_2(F_*(\mathbf{x}_v(u, v)))\\ &= \phi_2(\cos v\, \mathbf{y}_v(u, \sin v))\\ &= \cos v\, \phi_2(\mathbf{y}_u(u, \sin v))\\ &= \cos v \times 1\\ &= \cos v.\end{aligned}$$

(b) By Theorem 5.7,

$$F^*(\phi_1 \wedge \phi_2) = (F^*\phi_1) \wedge (F^*\phi_2).$$

If we use this and the definition of wedge product, we obtain

$$\begin{aligned}F^*(\phi_1 \wedge \phi_2)(\mathbf{x}_u, \mathbf{x}_v) &= (F^*\phi_1)(\mathbf{x}_u) \times (F^*\phi_2)(\mathbf{x}_v)\\ &\quad - (F^*\phi_1)(\mathbf{x}_v) \times (F^*\phi_2)(\mathbf{x}_u)\\ &= 1 \times \cos v - 0 \times 0\\ &= \cos v.\end{aligned}$$

Note: We have dropped the parameters wherever it seemed safe to do so. This is usually where there is no doubt as to their value. When there are several sets of parameter values around, we retain them in all expressions.

Solution 5.2

This solution is an example of where it is a good idea to retain parameters because there are two sets around:

$$(u, v)$$

and

$$(u + v, u - v).$$

We start by calculating an explicit expression for U. As usual, the first things that we need are the partial velocities.

$$\begin{aligned}\mathbf{x}_u(u, v) &= (1, 0, v),\\ \mathbf{x}_v(u, v) &= (0, 1, u).\end{aligned}$$

It follows that

$$\begin{aligned}\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) &= (-v, -u, 1),\\ \|\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)\| &= \sqrt{1 + u^2 + v^2}.\end{aligned}$$

Thus

$$U(\mathbf{x}(u, v)) = \frac{(-v, -u, 1)}{\sqrt{1 + u^2 + v^2}}.$$

We now apply the definition of η.

$$\begin{aligned}\eta(\mathbf{x}_u(u, v), \mathbf{x}_v(u, v)) &= \mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) \cdot U(\mathbf{x}(u, v))\\ &= (-v, -u, 1) \cdot U(\mathbf{x}(u, v))\\ &= \frac{1 + u^2 + v^2}{\sqrt{1 + u^2 + v^2}}\\ &= \sqrt{1 + u^2 + v^2}.\end{aligned}$$

In order to find $F^*\eta$, we shall need $F_*(\mathbf{x}_u)$ and $F_*(\mathbf{x}_v)$. Using partial differentiation and the chain rule, we obtain the following.

$$\begin{aligned}F_*(\mathbf{x}_u(u, v)) &= \frac{\partial}{\partial u} F(\mathbf{x}(u, v))\\ &= \frac{\partial}{\partial u} \mathbf{x}(u + v, u - v)\\ &= \mathbf{x}_u(u + v, u - v)\frac{\partial(u + v)}{\partial u}\\ &\quad + \mathbf{x}_v(u + v, u - v)\frac{\partial(u - v)}{\partial u}\\ &= \mathbf{x}_u(u + v, u - v) + \mathbf{x}_v(u + v, u - v).\\ F_*(\mathbf{x}_v(u, v)) &= \frac{\partial}{\partial v} F(\mathbf{x}(u, v))\\ &= \frac{\partial}{\partial v} \mathbf{x}(u + v, u - v)\\ &= \mathbf{x}_u(u + v, u - v)\frac{\partial(u + v)}{\partial v}\\ &\quad + \mathbf{x}_v(u + v, u - v)\frac{\partial(u - v)}{\partial v}\\ &= \mathbf{x}_u(u + v, u - v) - \mathbf{x}_v(u + v, u - v).\end{aligned}$$

To complete the solution we use the general result

$$\eta(a\mathbf{x}_u + b\mathbf{x}_v, c\mathbf{x}_u + d\mathbf{x}_v) = (ad - bc)\eta(\mathbf{x}_u, \mathbf{x}_v).$$

We also observe that the expression obtained for η means

$$\eta(\mathbf{x}_u, \mathbf{x}_v) = \sqrt{1 + (\text{1st parameter})^2 + (\text{2nd parameter})^2}.$$

Since

$$(F^*\eta)(\mathbf{x}_u(u, v), \mathbf{x}_v(u, v)) = \eta(F_*(\mathbf{x}_u(u, v)), F_*(\mathbf{x}_v(u, v)))$$

and we already have expressions for

$$F_*(\mathbf{x}_u) \text{ and } F_*(\mathbf{x}_v),$$

we can put everything together to obtain as follows.

$$\begin{aligned}
&(F^*\eta)(\mathbf{x}_u(u,v), \mathbf{x}_v(u,v))\\
&\quad = \eta(F_*(\mathbf{x}_u(u,v)), F_*(\mathbf{x}_v(u,v)))\\
&\quad = \eta(\mathbf{x}_u + \mathbf{x}_v, \mathbf{x}_u - \mathbf{x}_v)\\
&\qquad \text{(evaluated at } (u+v, u-v))\\
&\quad = (1 \times (-1) - 1 \times 1)\eta(\mathbf{x}_u(u+v, u-v), \mathbf{x}_v(u+v, u-v))\\
&\quad = -2\eta(\mathbf{x}_u(u+v, u-v), \mathbf{x}_v(u+v, u-v))\\
&\quad = -2\sqrt{1 + (u+v)^2 + (u-v)^2}\\
&\quad = -2\sqrt{1 + 2u^2 + 2v^2}.
\end{aligned}$$

Note that we have used the result

$$\eta(a\mathbf{x}_u + b\mathbf{x}_v, c\mathbf{x}_u + d\mathbf{x}_v) = (ad - bc)\eta(\mathbf{x}_u, \mathbf{x}_v).$$

Solution 6.1

In the long run it is probably easiest to give an explicit expression for $\mathbf{x}$.

$$\mathbf{x} = ((1 + v\cos(u/2))\cos u, (1 + v\cos(u/2))\sin u, v\sin(u/2)).$$

(a) The partial velocities are

$$\begin{aligned}
\mathbf{x}_u &= (-(1 + v\cos(u/2))\sin u - (v/2)\sin(u/2)\cos u,\\
&\qquad (1 + v\cos(u/2))\cos u - (v/2)\sin(u/2)\sin u,\\
&\qquad (v/2)\cos(u/2));\\
\mathbf{x}_v &= (\cos(u/2)\cos u, \cos(u/2)\sin u, \sin(u/2)).
\end{aligned}$$

The easiest check for regularity is probably that given by calculating the expression $EG - F^2$. Careful algebra yields

$$\begin{aligned}
E &= \mathbf{x}_u \cdot \mathbf{x}_u\\
&= (1 + v\cos(u/2))^2(\cos^2 u + \sin^2 u) + (v^2/4)\\
&= (1 + v\cos(u/2))^2 + (v^2/4);\\
F &= \mathbf{x}_u \cdot \mathbf{x}_v\\
&= 0;\\
G &= \mathbf{x}_v \cdot \mathbf{x}_v\\
&= \cos^2(u/2)(\cos^2 u + \sin^2 u) + \sin^2(u/2)\\
&= 1.
\end{aligned}$$

Since $E \geq 1$, it follows that $EG - F^2 \neq 0$ and $\mathbf{x}$ is regular.

(b) We need a value of u_0 such that

$$\cos u_0 = 1, \ \sin u_0 = 0.$$

The easiest non-zero value is $u_0 = 2\pi$.

We have

$$\begin{aligned}
\mathbf{x}(0,0) &= \mathbf{x}(2\pi, 0) = (1,0,0);\\
\mathbf{x}_u(0,0) &= (0,1,0),\\
\mathbf{x}_u(2\pi,0) &= (0,1,0),\\
\mathbf{x}_v(0,0) &= (1,0,0),\\
\mathbf{x}_v(2\pi,0) &= (-1,0,0).
\end{aligned}$$

It follows that

$$\mathbf{x}_u(0,0) \times \mathbf{x}_v(0,0) = -\mathbf{x}_u(2\pi,0) \times \mathbf{x}_v(2\pi,0).$$

(c) All the coordinate functions that appear in the definition of f are differentiable and are combined in

$$f(\mathbf{x}(u,v)) = \mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v) \cdot Z(\mathbf{x}(u,v))$$

using only multiplication and addition. It follows that f is differentiable and hence continuous.

(d) We can regard $f(\mathbf{x}(u,0))$ as a continuous function $g(u)$. However,

$$\begin{aligned}
g(0) &= f(\mathbf{x}(0,0))\\
&= \mathbf{x}_u(0,0) \times \mathbf{x}_v(0,0) \cdot Z(\mathbf{x}(0,0))
\end{aligned}$$

and

$$\begin{aligned}
g(2\pi) &= f(\mathbf{x}(2\pi,0))\\
&= \mathbf{x}_u(2\pi,0) \times \mathbf{x}_v(2\pi,0) \cdot Z(\mathbf{x}(2\pi,0))\\
&= -\mathbf{x}_u(0,0) \times \mathbf{x}_v(0,0) \cdot Z(\mathbf{x}(0,0)).
\end{aligned}$$

The last line follows because $\mathbf{x}(0,0) = \mathbf{x}(2\pi,0)$ and just $\mathbf{x}_v$ changes sign.

It follows that, because g changes sign, it must vanish on $v = 0$.

However, g is defined as a dot product. The first term is

$$\mathbf{x}_u \times \mathbf{x}_v \neq 0,$$

because $\mathbf{x}$ is regular. It follows that, since

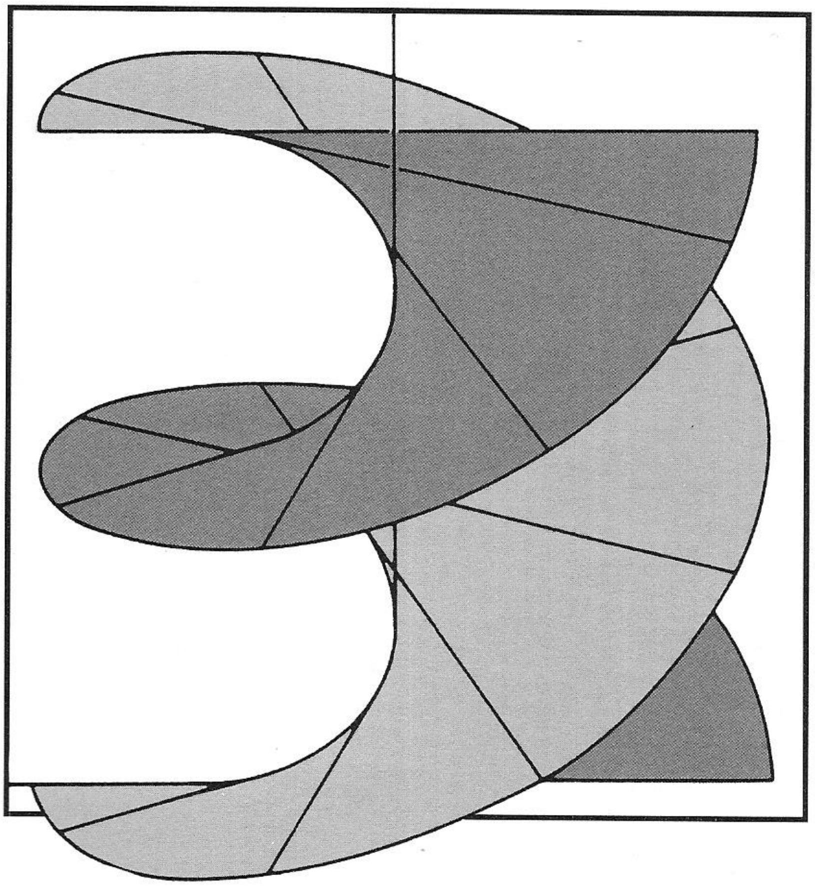
$$\mathbf{x}_u \times \mathbf{x}_v$$

is normal to the surface (and cannot be perpendicular to Z), that Z must vanish. This contradiction shows that M is not orientable because any attempt to define a non-vanishing normal vector field on M must fail.

ISBN 0 7492 4773 8

M434 Part V
Mathematics: A Fourth Level Course

M434
DIFFERENTIAL GEOMETRY



PART V
SHAPE OPERATORS

The Open University

The Open University

Mathematics: A Fourth Level Course

M434 Differential Geometry

Part V Shape Operators

Prepared for the Course Team
by Bob Margolis

Set book

Barrett O'Neill, *Elementary Differential Geometry*, hardback edition (Academic Press, 1966).

It is essential to have this book; the course is based on it and will not make sense without it.

The set book is referred to as *O'Neill*.

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published as *M334* 1976.

First published as *M434* 1993. Reprinted 1996.

Copyright © 1993 The Open University

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, without written permission from the publisher or a licence from the Copyright Licensing Agency Limited. Details of such licences (for reprographic reproduction) may be obtained from the Copyright Licensing Agency Ltd of 90 Tottenham Court Road, London, W1P 9HE.

Edited, designed and typeset by the Open University using the Open University TEX System.

Printed in the United Kingdom by Hobbs the Printers Ltd, Totton, Hampshire.

ISBN 0 7492 4774 6

This text forms part of an Open University Fourth Level Course. If you would like a copy of *Studying with The Open University*, please write to the Central Enquiry Service, PO Box 200, The Open University, Walton Hall, Milton Keynes, MK7 6YZ. If you have not already enrolled on the Course and would like to buy this or other Open University material, please write to Open University Educational Enterprises Ltd, 12 Cofferidge Close, Stony Stratford, Milton Keynes, MK11 1BY, United Kingdom.

1.2

Contents

Introduction 4

1 The shape operator 5

2 Normal curvature 9

3 Gaussian curvature 13

4 Computational techniques 16

5 Special curves 20

6 Surfaces of revolution 23

7 Summary 24

Solutions to the exercises 25

Introduction

Having developed the computational tools required, we can now apply them to the study of the 'shape' of surfaces in $\mathbf{E}^3$.

Section 1 is concerned with a definition of 'shape' and makes use of the idea of unit normal vector fields.

The idea behind using normal fields is, we think, a fairly natural one. Imagine living on the surface of a cylinder and carrying a unit normal with you from place to place. If you move along one of the generating lines then the unit normal will always point in the same direction—its rate of change is zero. However, moving along one of the cross-section circles causes the unit normal to rotate steadily.

That is, parallel to the axis of the cylinder.

The rate of change of the normal measures, in some sense, the shape of the surface along the path that you follow on the surface.

Section 2 makes use of the definitions of Section 1 to define curvature for surfaces. Since, as the cylinder example suggests, you would expect to experience different curvatures when moving in different directions, the curvature is a function not only of position on the surface, but also of direction. This section also provides some computational techniques.

Section 3 introduces two measurements of curvature that summarize the shape of a surface at each point. They are functions and, therefore, lose the information about differences in curvature in different directions. However, they are useful general descriptions.

The main purpose of Section 4 is to provide a range of computational techniques based on the partial velocities and their derivatives. In a sense the formulas derived correspond to those for curves that were expressed in terms of α', α'' and α'''.

The functions

$$E = \mathbf{x}_u \cdot \mathbf{x}_u, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v \quad \text{and} \quad G = \mathbf{x}_v \cdot \mathbf{x}_v,$$

which you have already met, play a central role in the computations.

Sections 5 and 6 apply the machinery that has been developed to two topics. The first is the discussion of particular types of curves in surfaces; the second is to surfaces of revolution.

Finally, Section 7 provides the usual summary.

Study advice

The following represents a possible plan for study weeks.

Week 1 *O'Neill*, Chapter V, Sections 1 and 2.
Week 2 *O'Neill*, Chapter V, Section 3 and TMA03.
Week 3 *O'Neill*, Chapter V, Sections 4 and 5.
Week 4 *O'Neill*, Chapter V, Section 6.

This leaves two study weeks for the work on three sections of Chapter VI and TMA04. Since the fourth week listed above is probably a little light, you may wish to start work on TMA04 during that week.

1 The shape operator

Read *O'Neill, Chapter V, Section 1, pages 189–193.*

The fundamental purpose of this section is to define how 'shape' is measured for *connected* surfaces in $\mathbf{E}^3$.

Connectedness is required so that the number of unit normal vector fields is limited. On the image of any patch there is always a choice of two unit normal vector fields: one in the same direction as

$$\mathbf{x}_u \times \mathbf{x}_v$$

at each point and the other in the opposite direction.

If a surface is orientable then there is a non-vanishing normal vector field which can be made unit length. The resulting unit normal, together with its negative, gives a choice of two unit normals on each connected component of the surface. If the surface has n connected components, then we can define 2^n unit normal vector fields.

We proved this result earlier.

Limiting ourselves to connected, orientable surfaces simplifies things somewhat. The theory developed can then be applied to each connected component of more general surfaces.

Covariant derivative *O'Neill* does not give a definition as such, although a definition is hinted at on page 189. The definition is

Definition 1.0

Let V be a vector field defined on a surface $M \subset \mathbf{E}^3$ and let $\mathbf{v_p}$ be a vector tangent to M. Then the covariant derivative $\nabla_{\mathbf{v_p}} V$ is defined as

$$V(\alpha(t))'(0),$$

where α is any curve in M such that

$$\alpha(0) = \mathbf{p},\ \alpha'(0) = \mathbf{v}.$$

This definition corresponds exactly to the definition of directional derivative on a surface.

Because of the importance of the parameter curves, we shall look at covariant derivatives with respect to the partial velocities.

The partial velocities form a basis for the tangent space at each point.

Suppose, then, that $M \subset \mathbf{E}^3$ is a surface and that $\mathbf{x}$ is a parametrization of M (or part of it). Let V be a vector field defined on the image of $\mathbf{x}$. We want to calculate

$$\nabla_{\mathbf{x}_u} V \quad \text{and} \quad \nabla_{\mathbf{x}_v} V.$$

We tackle them in turn.

The definition requires a curve with velocity $\mathbf{x}_u$, the obvious one to take is the u-parameter curve. To be quite explicit, we shall calculate

$$\nabla_{\mathbf{x}_u} V$$

at the point

$$\mathbf{p} = \mathbf{x}(u_0, v_0).$$

We define α by

$$\alpha(t) = \mathbf{x}(u_0 + t, v_0).$$

Since

$$\begin{aligned}\alpha'(t) &= \mathbf{x}_u(u_0+t,v_0)\times 1+\mathbf{x}_v(u_0+t,v_0)\times 0 \quad \text{(chain rule)}\\ &= \mathbf{x}_u(u_0+t,v_0),\end{aligned}$$

we have

$$\alpha'(0)=\mathbf{x}_u(u_0,v_0) \quad \text{and also} \quad \alpha(0)=\mathbf{x}(u_0,v_0)=\mathbf{p}.$$

Thus α is a suitable curve to use in the definition of covariant derivative. The definition then gives

$$\nabla_{\mathbf{x}_u(u_0,v_0)}V=V(\alpha(t))'(0).$$

Now $V(\alpha(t))=V(\mathbf{x}(u_0+t,v_0))$ and the chain rule then gives

$$\begin{aligned}V(\alpha(t))' &= \frac{\partial V(\mathbf{x}(u,v))}{\partial u}(u_0+t,v_0)\times 1+\frac{\partial V(\mathbf{x}(u,v))}{\partial v}(u_0+t,v_0)\times 0\\ &= \frac{\partial V(\mathbf{x})}{\partial u}(u_0+t,v_0).\end{aligned}$$

Evaluating at $t=0$ shows that

$$\nabla_{\mathbf{x}_u(u_0,v_0)}V=\frac{\partial V(\mathbf{x}(u,v))}{\partial u}(u_0,v_0).$$

If we express this result in 'function' form, we have

$$\nabla_{\mathbf{x}_u}V=\frac{\partial V(\mathbf{x})}{\partial u}.$$

Exactly similar arguments show that

$$\nabla_{\mathbf{x}_v}V=\frac{\partial V(\mathbf{x})}{\partial v}.$$

These two results are of immense importance for doing computations. They also provide further confirmation of the principle discussed earlier: that *all* forms of directional derivatives with respect to the partial velocities reduce to $\partial/\partial u$ and $\partial/\partial v$.

Shape operator The footnote about the reason for the minus sign is questionable, if only because there is always a choice of sign for the unit normal!

Note: The definition of shape operator can be extended to act on vector fields (rather than tangent vectors) by the usual pointwise process:

$$S(V):\mathbf{p}\longmapsto S(V(\mathbf{p})).$$

There is a link between the definition of shape operator and the connection forms discussed in *Part II*. There we summarized the connection equations in matrix form as

$$\nabla_V\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix}=\begin{pmatrix}0&\omega_{12}&\omega_{13}\\-\omega_{12}&0&\omega_{23}\\-\omega_{13}&-\omega_{23}&0\end{pmatrix}(V)\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix}$$

for any frame field E_1, E_2 and E_3.

Writing out the last line in full gives

$$\nabla_V E_3=-\omega_{13}E_1-\omega_{23}E_2.$$

Now suppose that we have a frame field defined on a surface such that E_1 and E_2 are tangent fields and $U=E_3$ is normal. It follows that the shape operator derived from U is given by

$$\begin{aligned}S(V)&=-\nabla_V U\\ &=\omega_{13}(V)E_1+\omega_{23}(V)E_2.\end{aligned}$$

This last equation seems a *good* reason for the introduction of the minus sign into the definition of S!

Using the connection forms is clumsier than working with the partial velocities *for surfaces in* $\mathbf{E}^3$. However, when we consider surfaces independent of $\mathbf{E}^3$, we have to abandon unit normals because they belong to the surrounding space, not to the surface. Then we shall have to make use of connection forms for tangent frame fields.

Example 1.3 In the exercises we ask you to verify, by direct calculation, the results discussed in the first and third of these examples. Here we consider the plane example.

Suppose that we have a parametrization

$$\mathbf{x} : (u, v) \longmapsto \mathbf{x}(u, v)$$

of a plane P in $\mathbf{E}^3$. Then, since the unit normal vector field U points in a constant direction, the coordinate functions of U must be independent of u and v. Hence

$$\begin{aligned} S(\mathbf{x}_u) &= -\nabla_{\mathbf{x}_u} U \\ &= -\frac{\partial U(\mathbf{x}(u,v))}{\partial u} = 0; \\ S(\mathbf{x}_v) &= -\nabla_{\mathbf{x}_v} U \\ &= -\frac{\partial U(\mathbf{x}(u,v))}{\partial v} = 0. \end{aligned}$$

Thus S is the zero operator at all points of P.

Remark: The appearance of the factor $1/r$ in the shape operator for the sphere should remind you of the appearance of $1/r$ as the curvature of a circle of radius r.

Symmetric operators You have probably met 2×2 symmetric matrices before. Their main property of interest then was that non-singular real, symmetric matrices have real eigenvalues and can always be diagonalized by an orthogonal matrix (actually by a rotation).

M101, Block IV.

The same is true of any matrix representing a non-singular, linear transformation which has the generalized symmetric property defined in Lemma 1.4.

Determinant and trace We shall use the trace and determinant of 2×2 matrices in Section 3. Here we want to mention some properties.

We shall be concerned with the matrices representing shape operators with respect to various bases. Since S is a linear transformation from a two-dimensional space to itself, all such matrices are 2×2. If A and B are the matrices with respect to two different bases, then they are related by

$$B = P^{-1}AP,$$

where P is non-singular.

It follows that

$$\begin{aligned} \det(B) &= \det(P^{-1}AP) \\ &= \det(P^{-1})\det(A)\det(P) \\ &= (\det(P))^{-1}\det(A)\det(P) \\ &= \det(A). \end{aligned}$$

Thus we may speak of the determinant of the shape operator, rather than of a particular matrix representing it.

The trace of a matrix is the sum of the elements on the leading diagonal and is denoted by trace(A).

What is not at all obvious is that, with the above notation,

$$\operatorname{trace}(B) = \operatorname{trace}(A).$$

(You can verify this by straightforward, but tedious, algebra.)

Thus determinant and trace are notions belonging to the shape operator, not to its representing matrices.

The matrix of a shape operator From time to time we shall want to write down the the matrix of a shape operator with respect to a particular basis of tangent vectors, usually the partial velocities.

So that there is no danger of writing down the transpose of the correct matrix, we discuss the details here.

Suppose that we have a basis of tangent vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ at a point and we have calculated the effect of S on them and expressed the results in the form

$$\begin{aligned} S(\mathbf{e}_1) &= a\mathbf{e}_1 + b\mathbf{e}_2, \\ S(\mathbf{e}_2) &= c\mathbf{e}_1 + d\mathbf{e}_2. \end{aligned}$$

Suppose that we have a tangent vector

$$\mathbf{v} = p\mathbf{e}_1 + q\mathbf{e}_2$$

at the same point. By the linearity of S, we have

$$\begin{aligned} S(\mathbf{v}) &= pS(\mathbf{e}_1) + qS(\mathbf{e}_2) \\ &= p(a\mathbf{e}_1 + b\mathbf{e}_2) + q(c\mathbf{e}_1 + d\mathbf{e}_2) \\ &= (pa + qc)\mathbf{e}_1 + (pb + qd)\mathbf{e}_2. \end{aligned}$$

Expressing the effect of S in terms of the coordinates with respect to $\mathbf{e}_1$ and $\mathbf{e}_2$, we have

$$\begin{pmatrix} p \\ q \end{pmatrix} \longmapsto \begin{pmatrix} pa + qc \\ pb + qd \end{pmatrix},$$

which can be achieved by the matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Note that the coefficients from the images of the basis form the *columns* of the matrix of S.

This is actually what you have seen before when constructing matrix transformations from the knowledge of the images of the basis

$$(1,0) \quad \text{and} \quad (0,1).$$

There is an important result that *O'Neill* leaves as Exercise 1 in this section. We shall want to make use of it, so we prove it here. You may wish to try the exercise before reading on.

Lemma 1.5

Let $M \subset \mathbf{E}^3$ be a surface and let α be a curve in M. If U is the restriction of a unit normal vector field on M to α, then

$$S(\alpha') = -U',$$

where S is the shape operator derived from U.

Proof We apply Definition 1.0 from earlier in this commentary. The details are a little messy if we are going to be precise.

Let $\alpha'(t_0)$ be the velocity at $\alpha(t_0)$ on α. Then the curve β, defined by

$$\beta(s) = \alpha(t_0 + s)$$

satisfies

$$\beta(0) = \alpha(t_0) \quad \text{and} \quad \beta'(0) = \alpha'(t_0).$$

By Definition 1.0, we can deduce that

$$\nabla_{\beta'(0)}U = (U(\beta(s)))'(0).$$

But $\beta'(0) = \alpha'(t_0)$ and

$$(U(\beta(s)))'(0) = (U(\alpha(t)))'(t_0),$$

so

$$\begin{aligned} S(\alpha')(t_0) &= -\nabla_{\beta'(0)}U \\ &= -(U(\alpha(t)))'(t_0). \end{aligned}$$

Since $(U(\alpha(t)))'$ is, by definition, U' and since the above calculation holds for all values of t_0, we can deduce that

$$S(\alpha') = -U'.$$

The complication in the proof is the need to introduce β in order to apply Definition 1.0 directly.

The following exercises are concerned with verifying some of the statements in Example 1.3. The saddle surface can be dealt with by similar means.

Exercise 1.1 Let M be the cylinder parametrized by

$$\mathbf{x}(u, v) = (r\cos u, r\sin u, v).$$

(a) Use the partial velocities to define an outward unit normal vector field, U on M.

(b) Let S be the shape operator on M derived from U. Find $S(\mathbf{x}_u(u, v))$ and $S(\mathbf{x}_v(u, v))$.

(c) Find the matrix representing S with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$.

Exercise 1.2 Let Σ be the sphere of radius r parametrized by

$$\mathbf{x}(u, v) = (r\cos u\cos v, r\cos u\sin v, r\sin u).$$

By following a scheme similar to that for the cylinder in the previous exercise, verify the assertion in the text that

$$S(\mathbf{v}) = -\frac{\mathbf{v}}{r}$$

on Σ.

[*Solutions on page 25*]

2 Normal curvature

Read *O'Neill, Chapter V, Section 2, pages 195–202.*

Erratum *O'Neill*, page 201, in Fig. 5.17, there is a bracket missing. The label for the point on the surface should read:

$$(x, y, f(x, y)). \quad \blacksquare$$

The main idea introduced in this section is *normal curvature*. This is the formalization of the notion discussed earlier that travelling in different directions on a surface causes you to experience different curvatures.

Lemma 2.1 This is an application of one manifestation of the Leibniz property. It will be of great practical importance for the computational techniques in Section 4. Note that heavy use is made of the result

$$S(\alpha') = -U'$$

that we proved at the end of the last section.

Definition 2.2 Note that the definition of k restricts its domain to *unit* tangent vectors to M.

The remarks following the definition link normal curvature to the (Frenet) curvature of curves in M.

The normal curvature function is somewhat unusual, for differential geometry, in that it is *not* linear. It cannot be, since it is defined only on unit vectors.

However, suppose that $\mathbf{v}_\mathrm{p}$ is non-zero and tangent to M. Then

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||}$$

is a unit vector and

$$\begin{aligned} k(\mathbf{u}) &= S(\mathbf{u}) \cdot \mathbf{u} \\ &= S\left(\frac{\mathbf{v}}{||\mathbf{v}||}\right) \cdot \left(\frac{\mathbf{v}}{||\mathbf{v}||}\right) \\ &= \frac{1}{||\mathbf{v}||^2} S(\mathbf{v}) \cdot (\mathbf{v}), \end{aligned}$$

by the linearity of S and dot products.

Thus

$$S(\mathbf{v}) \cdot \mathbf{v} = ||\mathbf{v}||^2 k(\mathbf{u}),$$

where $\mathbf{u}$ is a unit vector in the direction of $\mathbf{v}$. This last equation can be used to find the normal curvature in the direction of a given non-unit tangent vector (often a partial velocity).

Example We look algebraically at the saddle surface example in the last paragraph of page 198.

The surface can be covered by the single Monge patch

$$\mathbf{x}(u, v) = (u, v, uv).$$

The partial velocities are

$$\mathbf{x}_u(u, v) = (1, 0, v) \quad \text{and} \quad \mathbf{x}_v(u, v) = (0, 1, u).$$

We can construct a unit normal vector field from the partial velocities. This process gives

$$\begin{aligned} U(\mathbf{x}(u, v)) &= \frac{\mathbf{x}_u \times \mathbf{x}_v}{||\mathbf{x}_u \times \mathbf{x}_v||} \\ &= \frac{(-v, -u, 1)}{\sqrt{1 + u^2 + v^2}}. \end{aligned}$$

In order to find normal curvatures we need to know about the shape operator. As usual, we first find its effect on the partial velocities by partial differentiation. The details are a little messy, but careful differentiation yields

$$\begin{aligned} \frac{\partial U}{\partial u} &= \frac{(uv, -(1 + v^2), -u)}{(1 + u^2 + v^2)^{3/2}} \\ \frac{\partial U}{\partial v} &= \frac{(-(1 + u^2), uv, -v)}{(1 + u^2 + v^2)^{3/2}}. \end{aligned}$$

It follows that

$$\begin{aligned} S(\mathbf{x}_u) &= -\frac{\partial U}{\partial u} \\ &= \frac{(-uv, 1+v^2, u)}{(1+u^2+v^2)^{3/2}}, \end{aligned}$$

$$\begin{aligned} S(\mathbf{x}_v) &= -\frac{\partial U}{\partial v} \\ &= \frac{(1+u^2, -uv, v)}{(1+u^2+v^2)^{3/2}}. \end{aligned}$$

In the special case discussed in *O'Neill*, we have $u = v = 0$. Here

$$\begin{aligned} \mathbf{x}_u(0,0) &= (1,0,0), \\ \mathbf{x}_v(0,0) &= (0,1,0), \\ S(\mathbf{x}_u) &= (0,1,0), \\ S(\mathbf{x}_v) &= (1,0,0). \end{aligned}$$

Because the partial velocities at the origin are unit length, we have

$$\begin{aligned} k(\mathbf{x}_u) &= S(\mathbf{x}_u) \cdot \mathbf{x}_u \\ &= (0,1,0) \cdot (1,0,0) \\ &= 0, \end{aligned}$$

$$\begin{aligned} k(\mathbf{x}_v) &= (1,0,0) \cdot (0,1,0) \\ &= 0. \end{aligned}$$

These agree with *O'Neill*.

Now consider the direction of the line $x = y$ in the plane $z = 0$. One vector in this direction is

$$\mathbf{v} = \mathbf{x}_u(0,0) + \mathbf{x}_v(0,0) = (1,1,0)$$

and a unit vector in the same direction is

$$\mathbf{u} = \frac{1}{\sqrt{2}}(1,1,0).$$

By linearity,

$$S(\mathbf{u}) = \frac{1}{\sqrt{2}}(S(\mathbf{x}_u) + S(\mathbf{x}_v)) = \frac{1}{\sqrt{2}}(1,1,0).$$

Thus

$$\begin{aligned} k(\mathbf{u}) &= \frac{1}{\sqrt{2}}(1,1,0) \cdot \frac{1}{\sqrt{2}}(1,1,0) \\ &= 1. \end{aligned}$$

We can look at the direction of $y = -x$ by considering $\mathbf{x}_u - \mathbf{x}_v$. A suitable unit vector is

$$\mathbf{w} = \frac{1}{\sqrt{2}}(1,-1,0),$$

leading to

$$k(\mathbf{w}) = -1.$$

Principal curvatures and directions In spite of the assertion about being able to 'pick out' principal directions, one of the course's aims is to have computational tools for finding principal curvatures and directions.

The proof of Theorem 2.5 is a first-principles proof of the eigenvalue and eigenvector properties of symmetric linear transformations on two-dimensional spaces.

Where we refer to eigenvalues and eigenvectors, *O'Neill* uses the terms *characteristic* values and *characteristic* vectors.

We can summarize the results as follows. At each point of the surface:

(a) there is an orthonormal basis of eigenvectors of S;

(b) the eigenvalues of S are the extreme values of the normal curvature at that point, that is, they are the principal curvatures at that point;

(c) the eigenvectors define the principal directions at that point.

These results tie the shape of the surface to the algebraic properties of the shape operator S.

It is sometimes possible to find the matrix representing S with respect to the partial velocities fairly easily. From this matrix, we can calculate the eigenvalues and eigenvectors of S quite simply.

Sometimes the calculation of the matrix is not straightforward.

We pursue the saddle surface example a little further. From our calculations earlier, we can deduce (after some working) that

$$S(\mathbf{x}_u) = \frac{1}{(1+u^2+v^2)^{3/2}}(-uv\mathbf{x}_u + (1+v^2)\mathbf{x}_v),$$

$$S(\mathbf{x}_v) = \frac{1}{(1+u^2+v^2)^{3/2}}((1+u^2)\mathbf{x}_u - uv\mathbf{x}_v).$$

The matrix of S with respect to the partial velocities is, therefore,

$$\begin{pmatrix} \dfrac{-uv}{(1+u^2+v^2)^{3/2}} & \dfrac{1+u^2}{(1+u^2+v^2)^{3/2}} \\[2ex] \dfrac{1+v^2}{(1+u^2+v^2)^{3/2}} & \dfrac{-uv}{(1+u^2+v^2)^{3/2}} \end{pmatrix}.$$

Here there is scope for writing down the transpose of the correct matrix. Working with the transpose would make no difference to the determinant, trace or eigenvalues but the eigenvector calculations would be incorrect

The general expressions for the eigenvalues of this matrix are far from simple! However, at $u = v = 0$, the matrix reduces to

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

which has eigenvalues 1 and -1. The usual method of finding eigenvectors yields $(1, 1)$ and $(1, -1)$ as the coordinates with respect to the partial velocities. Thus the principal directions at the origin are in the directions of

$$\mathbf{x}_u(0,0) \pm \mathbf{x}_v(0,0),$$

which agrees with those we obtained earlier.

Umbilic points Definition 2.4 and Theorem 2.5 can be combined to give a practical test for umbilic points.

The function

$$k(\mathbf{u}) = k_1 \cos^2\vartheta + k_2 \sin^2\vartheta,$$

of Corollary 2.6, is a continuous function of ϑ. Because the eigenvalues k_1 and k_2 of the shape operator are the maximum and minimum values of a continuous function, we can deduce that $k(\mathbf{u})$ is constant if, and only if,

$$k_1 = k_2.$$

Now, if we have a matrix for S, we can express the condition for the eigenvalue equation to have equal roots. Suppose that the matrix is

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Then the eigenvalue equation reduces to

$$k^2 - (a+d)k + (ad - bc) = 0.$$

It is rather more revealing to write this in the form

$$k^2 - \operatorname{trace}(A)\,k + \det(A) = 0.$$

The quadratic equation formula gives '$b^2 = 4ac$' as the condition for equal roots. In the case of the above eigenvalue equation, this becomes

$$(\mathrm{trace}(A))^2 = 4 \times \det(A).$$

Let us apply this test to the saddle surface example. From the matrix

$$A = \begin{pmatrix} \dfrac{-uv}{(1+u^2+v^2)^{3/2}} & \dfrac{1+u^2}{(1+u^2+v^2)^{3/2}} \\ \\ \dfrac{1+v^2}{(1+u^2+v^2)^{3/2}} & \dfrac{-uv}{(1+u^2+v^2)^{3/2}} \end{pmatrix},$$

we have

$$\mathrm{trace}(A) = \frac{-2uv}{(1+u^2+v^2)^{3/2}},$$

$$\det(A) = \frac{u^2v^2 - (1+u^2)(1+v^2)}{(1+u^2+v^2)^3}.$$

Thus, a point with parameters (u, v) will be umbilic if, and only if,

$$\left(\frac{-2uv}{(1+u^2+v^2)^{3/2}}\right)^2 = 4\left(\frac{u^2v^2 - (1+u^2)(1+v^2)}{(1+u^2+v^2)^3}\right).$$

Both sides have the same denominator so the condition reduces to

$$u^2v^2 = u^2v^2 - (1+u^2)(1+v^2).$$

Since the second term on the right-hand side must be at least 1, there can be no solutions. We deduce that the saddle surface has no umbilic points.

Exercise 2.1 This question concerns the cylinder M parametrized by the mapping

$$\mathbf{x}(u, v) = (r\cos u, r\sin u, v), \qquad r > 0.$$

It is a continuation of Exercise 1.1 using the notation and results of that exercise.

(a) Find the eigenvalues and eigenvectors of the shape operator S and, hence, the principal curvatures and directions.

(b) Does M have any umbilic points?

(c) Find the normal curvature in the direction of $\mathbf{x}_u + \mathbf{x}_v$.

Exercise 2.2 *O'Neill*, page 202, Exercise 3. Note that the curves given can be defined by $\alpha(t) = \mathbf{x}(t, \pm t^n)$, where $\mathbf{x}$ is the parametrization in Exercise 2.1.

[*Solutions on page 25*]

3 Gaussian curvature

Read *O'Neill, Chapter V, Section 3, pages 203–207.*

Erratum *O'Neill*, page 204, ninth line and last but one line; page 205, four lines from the bottom of the page (first line of the proof) there are occurrences where:

for $T_{\mathbf{p}}(M)$ read $\overline{T}_{\mathbf{p}}(M)$. ■

This section introduces two new definitions, discusses some examples and proves the results that will be the basis of some straightforward computational techniques.

Gaussian and mean curvature We remarked earlier that the determinant and trace were the same for all matrices representing a shape operator. Hence Gaussian and mean curvatures are well defined.

Note that both K and H are functions from the surface to $\mathbf{R}$.

In the examples *O'Neill* indicates how the sign of K can be deduced by knowing the shape (in the intuitive sense) of the surface. Once the link between the formal definitions and intuition is established, we usually calculate the K and H functions to *provide* information about a new surface.

Examples As *O'Neill* says, we shall check a number of the results in these examples once we have efficient methods of calculation.

In *theory* what we have already would enable us to calculate K and H for, say, the monkey saddle parametrized by

$$\begin{aligned}\mathbf{x}(u,v) &= (u, v, u(u+\sqrt{3}v)(u-\sqrt{3}v)) \\ &= (u, v, u(u^2 - 3v^2)).\end{aligned}$$

In practice, we run into the same difficulties as we did when trying to apply directly the Frenet formulas for non-unit speed curves: some of the differentiation becomes so involved as to invite errors.

To illustrate this point, consider the monkey saddle parametrization above. We can find the partial velocities:

$$\begin{aligned}\mathbf{x}_u &= (1, 0, 3u^2 - 3v^2) \\ &= (1, 0, 3(u^2 - v^2)), \\ \mathbf{x}_v &= (0, 1, -6uv).\end{aligned}$$

If we construct a unit normal vector field from these, we have

$$\begin{aligned}\mathbf{x}_u \times \mathbf{x}_v &= (-3(u^2 - v^2), 6uv, 1), \\ \|\mathbf{x}_u \times \mathbf{x}_v\|^2 &= 9(u^2 - v^2)^2 + 36u^2v^2 + 1 \\ &= 9(u^2 + v^2)^2 + 1\end{aligned}$$

and so

$$U(\mathbf{x}(u,v)) = \frac{(-3(u^2 - v^2), 6uv, 1)}{\sqrt{9(u^2 + v^2)^2 + 1}}.$$

Now, finding $S(\mathbf{x}_u)$ etc. requires the partial derivatives of U. These will be involved expressions and it is unlikely to be easy to find the matrix representation of S with respect to $\mathbf{x}_u$ and $\mathbf{x}_v$.

The way out of such problems is similar to that for curves. We shall obtain formulas using higher derivatives of the parametrization.

Lemma 3.4 This result and the remarks following it provide the foundation for easier calculations. We shall apply it to the usual special case, where $\mathbf{v}$ and $\mathbf{w}$ are the partial velocities.

As Lemma 3.4 stands, it is no more use than the attempt above, since $S(\mathbf{v})$ etc. still appear, requiring partial differentiation of U. However, as so often, the Leibniz property will enable us to simplify things.

This simplification is in the next section.

The formulas after the end of the proof of Lemma 3.4 will appear in more memorable form later on.

Gaussian curvature and connection forms In Section 1, we discussed the link between the shape operator and the connection forms of a frame field made up of two unit tangent vector fields E_1 and E_2, together with a unit normal vector field

$$U = E_1 \times E_2.$$

We obtained

$$\begin{aligned} S(V) &= -\nabla_V U \\ &= \omega_{13}(V)E_1 + \omega_{23}(V)E_2, \end{aligned}$$

for any vector field V on the surface.

If we apply this to E_1 and E_2, we obtain

$$\begin{aligned} S(E_1) &= \omega_{13}(E_1)E_1 + \omega_{23}(E_1)E_2, \\ S(E_2) &= \omega_{13}(E_2)E_1 + \omega_{23}(E_2)E_2. \end{aligned}$$

The matrix of S with respect to E_1, E_2 is, therefore,

$$\begin{pmatrix} \omega_{13}(E_1) & \omega_{13}(E_2) \\ \omega_{23}(E_1) & \omega_{23}(E_2) \end{pmatrix},$$

with determinant

$$\omega_{13}(E_1)\omega_{23}(E_2) - \omega_{13}(E_2)\omega_{23}(E_1) = (\omega_{13} \wedge \omega_{23})(E_1, E_2).$$

We can obtain a further link by considering the second structural equations. In matrix form they become

Part II, Section 8.

$$d\omega = \omega \wedge \omega.$$

If we write out the $d\omega_{12}$ entry in full, we have

This is the second entry of the first row of the product.

$$\begin{aligned} d\omega_{12} &= \omega_{11} \wedge \omega_{12} + \omega_{12} \wedge \omega_{22} + \omega_{13} \wedge \omega_{32} \\ &= 0 \wedge \omega_{12} + \omega_{12} \wedge 0 - \omega_{13} \wedge \omega_{23} \\ &= -\omega_{13} \wedge \omega_{23}. \end{aligned}$$

Combining this with our earlier calculations, we see that

$$\begin{aligned} d\omega_{12}(E_1, E_2) &= -(\omega_{13} \wedge \omega_{23})(E_1, E_2) \\ &= -K. \end{aligned}$$

Thus

$$K = -d\omega_{12}(E_1, E_2).$$

This last equation should be interpreted as

$$K(\mathbf{p}) = -d\omega_{12}(E_1(\mathbf{p}), E_2(\mathbf{p})),$$

for each point $\mathbf{p}$ on the surface.

Thus, if we can find independent means of calculating connection forms on a surface, then we can calculate Gaussian curvature.

We shall pursue this line in *Part VI*.

Exercise 3.1 *O'Neill*, page 207, Exercise 1.

Exercise 3.2 Let M be the saddle surface parametrized by

$$\mathbf{x}(u, v) = (u, v, uv).$$

Use the matrix of the shape operator found in Section 2 to find the Gaussian and mean curvatures.

Does M have any planar points?

Exercise 3.3 *O'Neill*, page 207, Exercise 3. (*Hint*: The notation of Section 2, Corollary 2.6 should prove useful.)

[*Solutions on page 26*]

4 Computational techniques

Read *O'Neill, Chapter V, Section 4, pages 210–219.*

Errata

1 *O'Neill*, page 210, in the third displayed line, the expression for $\mathbf{w}$ should read:

$$\mathbf{w} = w_1\mathbf{x}_u + w_2\mathbf{x}_v.$$

2 *O'Neill*, page 214, in the final display line, the inequalities for K should read:

$$-\frac{1}{b^2} \leq K < 0.$$

3 *O'Neill*, page 218, in the last paragraph and in the diagram, the two occurrences of $T_\mathbf{p}(M)$ should read:

$$\overline{T}_\mathbf{p}(M).$$

■

The main aim of this section is to provide computationally efficient formulas for Gaussian and mean curvature. We shall use the partial velocities to build up such formulas.

Precision and convenience Near the beginning of the reading passage *O'Neill* adopts a rather more precise approach to exactly what the partial velocities are. It is worth looking hard at what he is saying and comparing it with what it is convenient to assume.

Let us look at a simple specific example to illustrate what the fuss is about. Consider the saddle surface M parametrized by the single proper patch $\mathbf{x}$ defined by

$$\mathbf{x}(u, v) = (u, v, uv), \quad (u, v) \in \mathbf{E}^2.$$

As we have observed much earlier, the partial velocities

$$\mathbf{x}_u(u, v) = (1, 0, v)$$
$$\mathbf{x}_v(u, v) = (0, 1, u)$$

are tangent vectors to M at the point $\mathbf{x}(u, v)$ and are the velocity vectors of the u-parameter and v-parameter curve passing through that point.

Thus, $\mathbf{x}_u$ and $\mathbf{x}_v$ are functions *from the domain of* $\mathbf{x}$ to the collection of all tangent vectors to M. The partial velocities are not, strictly, vector fields on M. Such vector fields have domain M, not $\mathbf{E}^2$.

That said, however, there are two vector fields that we can define on M:

$$\mathbf{x}(u, v) \longmapsto \mathbf{x}_u(u, v),$$
$$\mathbf{x}(u, v) \longmapsto \mathbf{x}_v(u, v).$$

There seems to be little harm in referring to these by the same names as the partial velocities, so long as we realize that it *is* an abuse of notation. (It is similar to the abuse involved in not distinguishing between f and $f(\mathbf{x})$ for a function on a surface.)

The discussion above generalizes from the saddle surface to any surface covered by a single patch. It also applies whenever there is a parametrization onto the whole surface. The case where it cannot automatically be applied is where the surface is defined by several overlapping patches. The difficulty is that the partial velocities of different, overlapping patches may well not agree on the overlap.

However, as discussed earlier, the unit normal defined as

Part IV, Section 6.

$$U = \frac{\mathbf{x}_u \times \mathbf{x}_v}{||\mathbf{x}_u \times \mathbf{x}_v||}$$

is well defined, even where patches overlap. There is a small abuse of notation in referring to U as a unit normal vector field on the surface, instead of a unit normal vector valued function on the domain of $\mathbf{x}$. The abuse amounts to referring to both

$$(u,v) \longmapsto \frac{\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)}{\|\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)\|}$$

and

$$\mathbf{x}(u,v) \longmapsto \frac{\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)}{\|\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)\|}$$

as U. Having mentioned that this *is* abuse, it seems fairly harmless to us!

Higher derivatives The notation $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$ and $\mathbf{x}_{vv}$ for the second derivatives is consistent with that for partial velocities.

The functions l, m and n The motivation for introducing these functions stems from the expressions for K and H (*O'Neill*, page 206). In those formulas, expressions such as

$$S(V)\cdot V,\ S(V)\cdot W \quad \text{and} \quad S(W)\cdot V$$

appear.

If we apply the formula for K (page 206) to the special case, where V is the vector field defined by $\mathbf{x}_u$ and W corresponds to $\mathbf{x}_v$, then we obtain

$$\begin{aligned} K &= \frac{\begin{vmatrix} S(\mathbf{x}_u)\cdot\mathbf{x}_u & S(\mathbf{x}_u)\cdot\mathbf{x}_v \\ S(\mathbf{x}_v)\cdot\mathbf{x}_u & S(\mathbf{x}_v)\cdot\mathbf{x}_v \end{vmatrix}}{\begin{vmatrix} \mathbf{x}_u\cdot\mathbf{x}_u & \mathbf{x}_u\cdot\mathbf{x}_v \\ \mathbf{x}_v\cdot\mathbf{x}_u & \mathbf{x}_v\cdot\mathbf{x}_v \end{vmatrix}} \\ &= \frac{\begin{vmatrix} l & m \\ m & n \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}} \\ &= \frac{ln - m^2}{EG - F^2}. \end{aligned}$$

Similar considerations give a formula for H.

These three new functions, together with E, F and G also appear naturally in a simplistic approach to finding matrices of shape operators, which we now indicate.

Suppose that S is the shape operator derived from a unit normal vector field U. Then, at any point on the surface, we have

$$\begin{aligned} S(\mathbf{x}_u) &= a\mathbf{x}_u + b\mathbf{x}_v, \\ S(\mathbf{x}_v) &= c\mathbf{x}_u + d\mathbf{x}_v, \end{aligned}$$

for suitable coefficients a, b, c, d. We can obtain enough simultaneous linear equations to find the coefficients if we take the dot product of both these equations with both partial velocities.

This gives the following equations.

$$\begin{aligned} S(\mathbf{x}_u)\cdot\mathbf{x}_u &= a\mathbf{x}_u\cdot\mathbf{x}_u + b\mathbf{x}_v\cdot\mathbf{x}_u, \\ S(\mathbf{x}_u)\cdot\mathbf{x}_v &= a\mathbf{x}_u\cdot\mathbf{x}_v + b\mathbf{x}_v\cdot\mathbf{x}_v, \\ S(\mathbf{x}_v)\cdot\mathbf{x}_u &= c\mathbf{x}_u\cdot\mathbf{x}_u + d\mathbf{x}_v\cdot\mathbf{x}_u, \\ S(\mathbf{x}_v)\cdot\mathbf{x}_v &= c\mathbf{x}_u\cdot\mathbf{x}_v + d\mathbf{x}_v\cdot\mathbf{x}_v. \end{aligned}$$

Using the definitions of the various functions, these equations become

$$\begin{aligned} l &= aE + bF, \\ m &= aF + bG, \\ m &= cE + dF, \\ n &= cF + dG. \end{aligned}$$

The usual methods for solving linear simultaneous equations give

$$a=\frac{Gl-Fm}{EG-F^2},$$
$$b=\frac{-Fl+Gm}{EG-F^2},$$
$$c=\frac{Gm-Fn}{EG-F^2},$$
$$d=\frac{-Fm+Gn}{EG-F^2}.$$

This gives the following, rather unwieldy, matrix for S.

$$\begin{pmatrix} \dfrac{Gl-Fm}{EG-F^2} & \dfrac{Gm-Fn}{EG-F^2} \\[2ex] \dfrac{-Fl+Gm}{EG-F^2} & \dfrac{-Fm+Gn}{EG-F^2} \end{pmatrix}.$$

Although the matrix is unwieldy, expressions for its trace and determinant are relatively straightforward.

$$\begin{aligned} \det(S) &= \frac{Gl-Fm}{EG-F^2}\frac{-Fm+Gn}{EG-F^2} - \frac{-Fl+Gm}{EG-F^2}\frac{Gm-Fn}{EG-F^2} \\ &= \frac{(ln-m^2)(EG-F^2)}{(EG-F^2)^2} \\ &= \frac{ln-m^2}{EG-F^2} \quad \text{(after some algebra)}. \end{aligned}$$

$$\begin{aligned} \operatorname{trace}(S) &= \frac{Gl-Fm}{EG-F^2} + \frac{-Fm+En}{EG-F^2} \\ &= \frac{Gl+En-2Fm}{EG-F^2}. \end{aligned}$$

This 'first principles' approach yields the same formulas as Corollary 4.1.

Lemma 4.2 This is the key to making the formulas of Corollary 4.1 usable by avoiding the need to calculate partial derivatives of U.

The proof of the lemma uses three applications of the Leibniz property.

$$0=\frac{\partial}{\partial u}(U\cdot \mathbf{x}_u) = U_u\cdot \mathbf{x}_u + U\cdot \mathbf{x}_{uu}$$
$$0=\frac{\partial}{\partial u}(U\cdot \mathbf{x}_v) = U_u\cdot \mathbf{x}_v + U\cdot \mathbf{x}_{uv}$$
$$0=\frac{\partial}{\partial v}(U\cdot \mathbf{x}_v) = U_v\cdot \mathbf{x}_v + U\cdot \mathbf{x}_{vv}$$

These are then interpreted by using $U_u = -S(\mathbf{x}_u)$ etc. to give

$$l = U\cdot \mathbf{x}_{uu}$$
$$m = U\cdot \mathbf{x}_{uv}$$
$$n = U\cdot \mathbf{x}_{vv}.$$

Example We shall now use these formulas and Lemma 4.2 to pursue the saddle surface example. The following provides a little more detail than is given in *O'Neill*.

We have already obtained the following.

$$\mathbf{x}_u = (1,0,v),$$
$$\mathbf{x}_v = (0,1,u),$$
$$U = \frac{(-v,-u,1)}{\sqrt{1+u^2+v^2}}.$$

It follows that

$$E = \mathbf{x}_u \cdot \mathbf{x}_u = 1 + v^2,$$

$$F = \mathbf{x}_u \cdot \mathbf{x}_v = uv,$$

$$G = \mathbf{x}_v \cdot \mathbf{x}_v = 1 + u^2;$$

$$EG - F^2 = (1 + v^2)(1 + u^2) - u^2v^2 = 1 + u^2 + v^2.$$

The second derivatives are

$$\mathbf{x}_{uu} = (0,0,0),$$
$$\mathbf{x}_{uv} = (0,0,1),$$
$$\mathbf{x}_{vv} = (0,0,0).$$

Hence

$$l = U \cdot \mathbf{x}_{uu} = 0,$$

$$m = U \cdot \mathbf{x}_{uv} = \frac{1}{\sqrt{1 + u^2 + v^2}},$$

$$n = U \cdot \mathbf{x}_{vv} = 0.$$

Applying the formulas, we have

$$K = \frac{ln - m^2}{EG - F^2} = \frac{0 - (1/(\sqrt{1 + u^2 + v^2}))^2}{1 + u^2 + v^2} = \frac{-1}{(1 + u^2 + v^2)^2};$$

$$H = \frac{Gl + En - 2Fm}{2(EG - F^2)} = \frac{0 + 0 - 2(uv/\sqrt{1 + u^2 + v^2})}{2(1 + u^2 + v^2)} = \frac{-uv}{(1 + u^2 + v^2)^{3/2}}.$$

These are the same as obtained previously by different methods.

This example illustrates the benefit of the new methods. Defining a unit normal vector field is still required but partial differentiation of this unit normal is replaced by differentiating the partial velocities, often a much easier task.

Should the principal curvatures be required, they can be found as solutions of Corollary 3.5

$$k^2 - 2Hk + K = 0.$$

Lemma 4.4 We shall not use this result as much as the previous ones but, for completeness, we indicate the 'routine' deduction from Lemma 3.4. We suspect that *O'Neill* omitted the proof because of the algebra involved!

Firstly, we need a unit normal vector field. This is provided by reducing Z to unit length:

$$U = \frac{Z}{\|Z\|}.$$

Now we express $S(V)$ and $S(W)$ in terms of Z.

$$\begin{aligned} S(V) &= -\nabla_V U \\ &= -\nabla_V \frac{Z}{\|Z\|} \\ &= -\left(\frac{1}{\|Z\|}\nabla_V Z + V\left[\frac{1}{\|Z\|}\right] Z\right) \end{aligned}$$

(by the Leibniz formula for covariant derivatives).

Similarly,

$$S(W) = -\left(\frac{1}{\|Z\|}\nabla_W Z + W\left[\frac{1}{\|Z\|}\right] Z\right).$$

Taking the cross product and omitting zeros, we have

$$S(V) \times S(W) = \frac{1}{\|Z\|^2}(\nabla_V Z \times \nabla_W Z) + \frac{1}{\|Z\|}\left(V\left[\frac{1}{\|Z\|^2}\right]\nabla_V Z \times Z + V\left[\frac{1}{\|Z\|^2}\right] Z \times \nabla_W Z\right).$$

When we take the dot product with Z, the last two terms disappear, so

$$Z \cdot (S(V) \times S(W)) = \frac{1}{\|Z\|^2} Z \cdot (\nabla_V Z \times \nabla_W Z).$$

From Lemma 3.4, we have

$$S(V) \times S(W) = KV \times W$$

so, taking the dot product with Z, we have

$$\begin{aligned} Z \cdot (S(V) \times S(W)) &= KZ \cdot V \times W \\ &= KZ \cdot Z \quad \text{(definition of } Z \text{ etc.)} \\ &= K\|Z\|^2. \end{aligned}$$

Combining with the previous result, we have

$$\frac{1}{\|Z\|^2} Z \cdot (\nabla_V Z \times \nabla_W Z) = K\|Z\|^2,$$

from which the first result follows.

The calculations for H are much the same and we omit them.

Example 4.5 This shows that applying Lemma 4.4 can require some ingenuity in the choice of V and W. We shall not make a great deal of use of this technique.

Patches and parametrizations The final remarks of this section of *O'Neill* show that calculations are much easier if we can find a single parametrization of the surface being studied. This will be the case in most of our work.

Exercise 4.1 *O'Neill*, page 219, Exercise 2. **Note:** Subscripts on f denote partial differentiation.

Exercise 4.2 *O'Neill*, page 219, Exercise 3.

Exercise 4.3 *O'Neill*, page 219, Exercise 4.

[*Solutions on page 27*]

5 Special curves

Read *O'Neill, Chapter V, Section 5, pages 223–229.*

Errata

1 *O'Neill*, page 224, the tenth line should read:

'...(unit) principal vectors belonging to ... .'

2 *O'Neill*, page 228, the second line should read:

'...$S(\alpha') = -U'$ ... .' ■

There is a sense in which two of the three types of curve studied here have definitions that follow naturally from what has gone before.

Principal and asymptotic curves.

We have singled out the principal directions at a point on a surface as being the directions where the normal curvature is extreme. It is not unreasonable, therefore, to consider curves that always point in a principal direction at each point on their routes.

Equally, directions in which the normal curvature is zero are of some interest since they can occur on surfaces which are most definitely curved in the intuitive sense.

The third type of curve—geodesics—arise from a different consideration: the search for the quickest way to get from one point to another of a surface *staying on the surface.*

Think about travelling around at constant *speed* on a surface. Because of the curvature of the surface some acceleration normal to the surface is more or less inevitable. However, intuitively, any acceleration tangent to the surface represents *avoidable* cornering. Thus, the quickest way to travel from one point to another is by a route whose acceleration is always normal to the surface.

The discussion above is intended as a brief motivation for the three definitions: 5.1, 5.5 and 5.7.

Together with the definitions, *O'Neill* proves results designed to give usable tests to see if a given curve is principal, asymptotic or a geodesic. In practice some of the tests are more usable than others.

Lemma 5.2 As a practical test, U' being collinear with α' suffers from the need to calculate U'. As we have seen for the saddle surface, this can be sufficiently involved to be error-inducing.

Asymptotic curves The final test in the paragraph following Definition 5.5 is more usable than that contained in Lemma 5.2 because it involves the second derivative of the curve rather than U'.

Geodesics The paragraph following the definition can be made to yield a specific test if we use the fact that the partial velocities of a patch (or parametrization) provide a basis for the tangent plane at each point. Therefore α is a geodesic if, and only if,

$$\alpha'' \cdot \mathbf{x}_u = \alpha'' \cdot \mathbf{x}_v = 0,$$

for all points on the route of α. This result is essentially stating that a vector is perpendicular to a plane precisely when it is orthogonal to a basis for the plane.

The tests are summarized on page 229, just before the exercises. Where there is more than one test, you only need show that *one* is satisfied, the tests in any one row of the table are equivalent.

General The search for special curves in a surface usually involves the solution of differential equations. To see why, consider a surface M parametrized by $\mathbf{x}$. Suppose that

$$\alpha = \mathbf{x}(\alpha_1, \alpha_2)$$

is a curve in M.

The chain rule enables α' and α'' to be expressed in terms of

$$\mathbf{x}_u,\ \mathbf{x}_v,\ \mathbf{x}_{uu},\ \mathbf{x}_{uv},\ \mathbf{x}_{vv}$$

and the derivatives of α_1 and α_2. Thus all the tests discussed in the section can be written in terms of differential equations satisfied by the coordinate functions α_1 and α_2.

Example As a simple example, let us apply some of these ideas to the information that we have already assembled about the saddle surface M parametrized by

$$\mathbf{x}(u,v) = (u, v, uv).$$

We consider a curve $\alpha = \mathbf{x}(\alpha_1, \alpha_2)$ in M and, rather than use the chain rule, we use the partial velocities and unit normal vector field already derived.

On the curve

$$\begin{aligned}
U(\alpha) &= \frac{(-\alpha_2, -\alpha_1, 1)}{\sqrt{1+\alpha_1^2+\alpha_2^2}},\\
\mathbf{x}_u(\alpha) &= (1, 0, \alpha_2),\\
\mathbf{x}_v(\alpha) &= (0, 1, \alpha_1),\\
\alpha' &= (\alpha_1', \alpha_2', \alpha_1'\alpha_2 + \alpha_1\alpha_2'),\\
\alpha'' &= (\alpha_1'', \alpha_2'', \alpha_1''\alpha_2 + 2\alpha_1'\alpha_2' + \alpha_1\alpha_2'').
\end{aligned}$$

We consider the case where α is asymptotic. Using the test

$$U \cdot \alpha'' = 0,$$

we obtain, after some algebra,

$$2\alpha_1'\alpha_2' = 0.$$

The regularity of α prevents both the derivatives being zero together. Hence the solutions are

$$\alpha_1 = \text{constant} \quad \text{or} \quad \alpha_2 = \text{constant}.$$

The asymptotic curves in M are, therefore, described by the condition that exactly one of the parameters must be held constant; that is, they are the parameter curves.

This result does correspond to the results obtained previously. We have shown that

$$l = U \cdot \mathbf{x}_{uu} = 0 \quad \text{and} \quad n = U \cdot \mathbf{x}_{vv} = 0.$$

But $\mathbf{x}_{uu}$ and $\mathbf{x}_{vv}$ are the accelerations of the parameter curves so

$$l = n = 0$$

says precisely that the parameter curves are asymptotic. ■

This example does illustrate the point about the appearance of differential equations. They are not usually as easy to solve!

Exercise 5.1 *O'Neill*, page 229, Exercise 1.

Exercise 5.2 *O'Neill*, page 230, Exercise 2. (*Hint*: You may wish to use the parametrization

$$\mathbf{x}(u,v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u), \qquad R > r > 0$$

for the torus. You may also wish to use the Monge patch for the saddle surface.)

Exercise 5.3 Let $M \subset \mathbf{E}^3$ be a surface and let α be a unit-speed asymptotic curve in M with positive curvature κ. Using the usual notation for the Frenet apparatus of α show that:

(a) B is normal to the surface along α;

(b) $S(T) = \tau N$.

[*Solutions on page 27*]

6 Surfaces of revolution

Read *O'Neill, Chapter V, Section 6, pages 235–242.*

Erratum *O'Neill*, page 241, five lines from the bottom, for the expressions given for k_μ and k_π, read:

$$k_\mu = \frac{h'}{c} \quad \text{and} \quad k_\pi = \frac{-1}{ch'}.$$

■

This section introduces no new ideas but provides some further examples of the use of the techniques discussed so far.

In a sense some of the work is 'general' in that it discusses the class of *all* surfaces of revolution. It is, however, at a lower level of generality than much of the theory.

On the whole *O'Neill* adopts the pragmatic approach to defining tangent vector fields and unit normal vector fields in terms of the partial velocities.

We have discussed the abuse of notation embodied in this approach at several points in the course.

Whilst it is useful to have formulas for the various curvatures of a general surface of revolution, in practice it is often simpler to deal with each case using the standard computational methods of Section 4.

One factor that appears in most of the formulas is the square of the speed of the profile curve:

$$E = g'^2 + h'^2.$$

If it is possible to arrange a unit-speed parametrization of the profile curve, then these formulas are simplified considerably.

Orthogonal parametrizations There is a more general point that arises from the discussion of surfaces of revolution.

A number of simplifications in the calculations arise because the partial velocities are orthogonal, that is,

$$F = \mathbf{x}_u \cdot \mathbf{x}_v = 0.$$

The most obvious simplification is in the term

$$EG - F^2$$

which appears in the formulas for both K and H.

A slightly less obvious one is that the partial velocities can be used to construct a *frame* field on the surface by using:

$$E_1 = \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} = \frac{\mathbf{x}_u}{\sqrt{E}},$$

$$E_2 = \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|} = \frac{\mathbf{x}_v}{\sqrt{G}},$$

$$U = E_1 \times E_2.$$

The benefit from this is that, should you want the matrix of the shape operator with respect to this basis, the coefficients can be obtained by applying orthonormal expansion to $S(E_1)$ and $S(E_2)$.

Note: It *is* true that the meridians and parallels of a surface of revolution are always principal, this is *not* true for general orthogonal parametrizations.

O'Neill, top of page 235.

Exercise 6.1 Let M be the surface of revolution obtained by rotating the 'half ellipse'

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad y>0, \quad a>b>0$$

about the x-axis.

(a) Use the parametrization of the profile curve

$$\alpha(u)=(a\cos u, b\sin u), \quad 0<u<\pi$$

to calculate the Gaussian curvature function on M.

(b) As defined, M is not a closed surface since the points $(\pm a, 0, 0)$ are missing. The parametrization may safely be extended to include these points for reasons that *O'Neill* discussed in an exercise in Chapter IV.

If we do this extension, what are the maximum and minimum values of the Gaussian curvature and where do these extreme values occur?

Exercise 6.2 Complete the investigation of surfaces of revolution with constant Gaussian curvature by considering the case $K=0$. The methods used in Examples 6.5 and 6.6 of *O'Neill* can be adapted to this case.

[*Solutions on page 28*]

7 Summary

Read *O'Neill, Chapter V, Section 7, page 244.*

We have made the basic definitions for measuring the shape of a surface and developed methods of computing the various functions.

The partial velocities have played a central role in this work. We have made very heavy use of the principle that all forms of directional derivative with respect to the partial velocities reduce to partial differentiation with respect to the parameters u and v.

The functions

$$E,\ F,\ G,\ l,\ m,\ n$$

have enabled us to prove the concise formulas

$$K=\frac{ln-m^2}{EG-F^2} \quad \text{and} \quad H=\frac{Gl+En-2Fm}{2(EG-F^2)}$$

for the Gaussian and mean curvature functions.

The principal curvatures can be calculated as the solutions of the quadratic equation

$$k^2-2Hk+K=0$$

to give

$$k_1, k_2 = H \pm \sqrt{H^2-K}.$$

We have also discussed the link between the shape operator as a linear transformation on each tangent plane and the various curvature functions, and that the determinant, eigenvectors and eigenvalues of the shape operators all have geometric significance.

Solutions to the exercises

Solution 1.1

(a) We calculate the partial derivatives and then use their cross product to define U. To save space, we suppress the parameters u and v.

$$\begin{aligned}\mathbf{x}_u &= (-r\sin u, r\cos u, 0),\\ \mathbf{x}_v &= (0,0,1),\\ \mathbf{x}_u \times \mathbf{x}_v &= (r\cos u, r\sin u, 0).\\ \|\mathbf{x}_u\times\mathbf{x}_v\|^2 &= r^2(\cos^2 u + \sin^2 u)\\ &= r^2.\\ U(\mathbf{x}(u,v)) &= (\cos u, \sin u, 0).\end{aligned}$$

There is only one other possible choice for U, the negative of the one given. However, as defined above, U does point outward from the surface as required.

(b) Applying the result in the commentary, we have

$$\begin{aligned}S(\mathbf{x}_u) &= -\nabla_{\mathbf{x}_u} U\\ &= -\frac{\partial U(\mathbf{x}(u,v))}{\partial u}\\ &= -\frac{\partial}{\partial u}(\cos u, \sin u, 0)\\ &= -(-\sin u, \cos u, 0)\\ &= (\sin u, -\cos u, 0).\end{aligned}$$

Similarly

$$\begin{aligned}S(\mathbf{x}_v) &= -\nabla_{\mathbf{x}_v} U\\ &= -\frac{\partial U(\mathbf{x}(u,v))}{\partial v}\\ &= -\frac{\partial}{\partial v}(\cos u, \sin u, 0)\\ &= (0,0,0).\end{aligned}$$

(c) From the calculations above, we have

$$\begin{aligned}S(\mathbf{x}_u) &= -\frac{1}{r}\mathbf{x}_u + 0\mathbf{x}_v,\\ S(\mathbf{x}_v) &= 0\mathbf{x}_u + 0\mathbf{x}_v.\end{aligned}$$

It follows that the matrix is

$$\begin{pmatrix} -\frac{1}{r} & 0\\ 0 & 0\end{pmatrix}.$$

In this case there is no harm done if you do transpose the matrix!

Solution 1.2

We follow the same steps as above, beginning with the partial velocities.

$$\begin{aligned}\mathbf{x}_u &= r(-\sin u\cos v, -\sin u\sin v, \cos u),\\ \mathbf{x}_v &= r(-\cos u\sin v, \cos u\cos v, 0),\\ \mathbf{x}_u\times\mathbf{x}_v &= r^2(-\cos^2 u\cos v, -\cos^2 u\sin v, -\sin u\cos u)\\ &= -r^2\cos u(\cos u\cos v, \cos u\sin v, \sin u).\end{aligned}$$

Note that the norm of the cross product is

$$r^2|\cos u|$$

since the vector part above is of unit length.

By comparing this cross product with the parametrization, we see that

$$U(\mathbf{x}(u,v)) = (\cos u\cos v, \cos u\sin v, \sin u)$$

defines an outward unit normal vector field on Σ.

Next we calculate the values of $S(\mathbf{x}_u)$, etc.

$$\begin{aligned}S(\mathbf{x}_u) &= -\nabla_{\mathbf{x}_u} U\\ &= -\frac{\partial U(\mathbf{x}(u,v))}{\partial u}\\ &= -\frac{\partial}{\partial u}(\cos u\cos v, \cos u\sin v, \sin u)\\ &= -(-\sin u\cos v, -\sin u\sin v, \cos u)\\ &= -\frac{1}{r}\mathbf{x}_u(u,v) + 0\mathbf{x}_v(u,v).\end{aligned}$$

Similarly,

$$\begin{aligned}S(\mathbf{x}_v) &= -\nabla_{\mathbf{x}_v} U\\ &= -\frac{\partial U(\mathbf{x}(u,v))}{\partial v}\\ &= -\frac{\partial}{\partial v}(\cos u\cos v, \cos u\sin v, \sin u)\\ &= -(-\cos u\sin v, \cos u\cos v, 0)\\ &= 0\mathbf{x}_u(u,v) - \frac{1}{r}\mathbf{x}_v(u,v).\end{aligned}$$

It follows that the matrix representing S with respect to the partial velocities is

$$\begin{pmatrix} -\frac{1}{r} & 0\\ 0 & -\frac{1}{r}\end{pmatrix}.$$

Now suppose that the tangent vector $\mathbf{v}$ to Σ is given by

$$\mathbf{v} = a\mathbf{x}_u + b\mathbf{x}_v.$$

Then, by linearity,

$$\begin{aligned}S(\mathbf{v}) &= S(a\mathbf{x}_u + b\mathbf{x}_v)\\ &= aS(\mathbf{x}_u) + bS(\mathbf{x}_v)\\ &= a\left(-\frac{1}{r}\right)\mathbf{x}_u + b\left(-\frac{1}{r}\right)\mathbf{x}_v\\ &= -\frac{1}{r}(a\mathbf{x}_u + b\mathbf{x}_v)\\ &= -\frac{\mathbf{v}}{r}.\end{aligned}$$

Solution 2.1

(a) From Exercise 1 of Section 1 we have that the matrix representing S with respect to the partial velocities is

$$\begin{pmatrix} -\frac{1}{r} & 0\\ 0 & 0\end{pmatrix}.$$

The eigenvalue equation is

$$k^2 + \frac{k}{r} = 0,$$

with solutions

$$k_1 = -\frac{1}{r} \quad\text{and}\quad k_2 = 0.$$

These are the principal curvatures.

It is clear from the matrix that

$$S(\mathbf{x}_u) = k_1\mathbf{x}_u \quad\text{and}\quad S(\mathbf{x}_v) = k_2\mathbf{x}_v.$$

Thus the partial velocities are eigenvectors of S and so they define the principal directions at each point.

(b) Since

$$-\frac{1}{r} \neq 0,$$

there can be no umbilic points on M.

(c) Suppose that $\mathbf{u}$ is a unit vector in the required direction. Since

$$\mathbf{x}_u + \mathbf{x}_v = (-r\cos u, r\sin u, 1)$$

and

$$\|\mathbf{x}_u + \mathbf{x}_v\| = \sqrt{1+r^2},$$

we have

$$\begin{aligned}
k(\mathbf{u}) &= \left(\frac{1}{\sqrt{1+r^2}}\right)^2 S(\mathbf{x}_u + \mathbf{x}_v)\cdot(\mathbf{x}_u + \mathbf{x}_v)\\
&= \frac{1}{1+r^2}\left(S(\mathbf{x}_u) + S(\mathbf{x}_v)\right)\cdot(\mathbf{x}_u + \mathbf{x}_v)\\
&\quad \text{(linearity of } S\text{)}\\
&= \frac{1}{1+r^2}\left(-\frac{1}{r}\mathbf{x}_u + 0\right)\cdot(\mathbf{x}_u + \mathbf{x}_v)\\
&= -\frac{1}{r(1+r^2)}\\
&\quad \times(-r\sin u, r\cos u, 0)\cdot(-r\sin u, r\cos u, 1)\\
&= -\frac{1}{r(1+r^2)}(r^2)\\
&= -\frac{r}{1+r^2}.
\end{aligned}$$

Solution 2.2

Note that $t=0$ corresponds to the point

$$\mathbf{x}(0,0)$$

of the cylinder.

We can use the unit normal already found above to deduce that the unit normal to the cylinder at $t=0$ is

$$U(\mathbf{x}(0,0)) = (1,0,0).$$

The acceleration of the curves can be calculated directly. The velocity and acceleration are

$$\begin{aligned}
\alpha'(t) &= (-r\sin t, r\cos t, \pm n t^{n-1}),\\
\alpha''(t) &= (-r\cos t, -r\sin t, \pm n(n-1)t^{n-2}).
\end{aligned}$$

Note that, for all such curves,

$$\alpha'(0) = (0, r, 0),$$

so the assertion about their velocities is correct.

We have

$$\alpha''(0) = (-r, 0, 0)$$

if $n > 2$ and

$$\alpha''(0) = (-r, 0, \pm 2)$$

if $n = 2$.

In either case

$$\alpha''(0)\cdot U(\mathbf{x}(0,0)) = -r.$$

This completes the proof of the assertion.

Solution 3.1

We use the notation of the section of *O'Neill*.

At an umbilic point, we have $k_1 = k_2$. But then

$$K = k_1 k_2 = k_1^2 \geq 0.$$

Hence, if $K < 0$ everywhere on the surface, there can be no umbilic points on the surface.

If $K \leq 0$ on the surface and we know that $K \geq 0$ at an umbilic point, at such a point we must have $K = 0$. But then

$$k_1 = k_2 = 0.$$

Thus any such umbilic point is planar.

Solution 3.2

The matrix obtained in Section 2 was

$$\begin{pmatrix}
\dfrac{-uv}{(1+u^2+v^2)^{3/2}} & \dfrac{1+u^2}{(1+u^2+v^2)^{3/2}}\\[2ex]
\dfrac{1+v^2}{(1+u^2+v^2)^{3/2}} & \dfrac{-uv}{(1+u^2+v^2)^{3/2}}
\end{pmatrix}.$$

This has determinant

$$\begin{aligned}
K &= \frac{u^2v^2 - (1+v^2)(1+u^2)}{(1+u^2+v^2)^3}\\
&= \frac{-(1+u^2+v^2)}{(1+u^2+v^2)^3}\\
&= -\frac{1}{(1+u^2+v^2)^2}
\end{aligned}$$

and half the trace is

$$H = \frac{-uv}{(1+u^2+v^2)^{3/2}}.$$

A planar point is one where $k_1 = k_2 = 0$ and hence $K = 0$. However, we have seen that

$$K = \frac{-1}{(1+u^2+v^2)^2} < 0$$

so there can be no planar points.

Solution 3.3

(a) Let $\mathbf{u}$ and $\mathbf{v}$ be two orthogonal unit tangent vectors at $\mathbf{p}$. In the notation of Corollary 2.6, these correspond to angles of, say, ϑ and $\vartheta + \pi/2$. Now

$$\begin{aligned}
\cos(\vartheta + \pi/2) &= \cos\vartheta\cos(\pi/2) - \sin\vartheta\sin(\pi/2)\\
&= -\sin\vartheta,\\
\sin(\vartheta + \pi/2) &= \sin\vartheta\cos(\pi/2) + \cos\vartheta\sin(\pi/2)\\
&= \cos\vartheta.
\end{aligned}$$

Hence

$$\begin{aligned}
k(\mathbf{u}) &= k_1\cos^2\vartheta + k_2\sin^2\vartheta,\\
k(\mathbf{v}) &= k_1\sin^2\vartheta + k_2\cos^2\vartheta.
\end{aligned}$$

If we add and use $\cos^2\vartheta + \sin^2\vartheta = 1$, we obtain

$$\begin{aligned}
k(\mathbf{u}) + k(\mathbf{v}) &= 2k_1 + 2k_2\\
&= 2H(\mathbf{p}).
\end{aligned}$$

The result follows.

Note that the above can also be used to justify the passing remark in the question:

$$\begin{aligned}
k(\mathbf{u})k(\mathbf{v}) &= (k_1^2 + k_2^2)\cos^2\vartheta\sin^2\vartheta\\
&\quad + k_1k_2(\cos^4\vartheta + \sin^4\vartheta) \neq K.
\end{aligned}$$

(b) We use

$$\cos 2\vartheta = 2\cos^2\vartheta - 1 = 1 - 2\sin^2\vartheta.$$

We have

$$\begin{aligned}
\int_0^{2\pi} k(\vartheta)\,d\vartheta &= \int_0^{2\pi}\left(k_1\cos^2\vartheta + k_2\sin^2\vartheta\right)d\vartheta\\
&= \int_0^{2\pi}\left(\frac{k_1}{2}(\cos 2\vartheta + 1) + \frac{k_2}{2}(1 - \cos 2\vartheta)\right)d\vartheta\\
&= \frac{k_1+k_2}{2}\int_0^{2\pi} d\vartheta + \frac{k_1-k_2}{2}\int_0^{2\pi}\cos 2\vartheta\,d\vartheta\\
&= \frac{k_1+k_2}{2}\times 2\pi + \frac{k_1-k_2}{2}\left[\frac{1}{2}\sin 2\vartheta\right]_0^{2\pi}\\
&= 2\pi H(\mathbf{p}).
\end{aligned}$$

Thus

$$\frac{1}{2\pi}\int_0^{2\pi} k(\vartheta)\,d\vartheta = H(\mathbf{p}).$$

Solution 4.1

This question requires the careful application of the various definitions and formulas. Firstly,

$$\mathbf{x}_u = (1, 0, f_u),$$
$$\mathbf{x}_v = (0, 1, f_v).$$

Thus

$$E = 1 + f_u^2,$$
$$F = f_u f_v,$$
$$G = 1 + f_v^2,$$

as asserted.

We can define a unit normal vector field, as usual, by

$$U = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}} = \frac{(-f_u, -f_v, 1)}{W}.$$

Next, we need l, m and n. Now

$$\mathbf{x}_{uu} = (0, 0, f_{uu}),$$
$$\mathbf{x}_{uv} = (0, 0, f_{uv}),$$
$$\mathbf{x}_{vv} = (0, 0, f_{vv}).$$

Hence

$$l = \frac{f_{uu}}{W},$$
$$m = \frac{f_{uv}}{W},$$
$$n = \frac{f_{vv}}{W},$$

as required.

The Gaussian and mean curvatures are calculated as follows.

$$K = \frac{ln - m^2}{EG - F^2} = \frac{(f_{uu} f_{vv} - f_{uv}^2)/W^2}{(1 + f_u^2)(1 + f_v^2) - f_u^2 f_v^2} = \frac{f_{uu} f_{vv} - f_{uv}^2}{W^2(1 + f_u^2 + f_v^2)} = \frac{f_{uu} f_{vv} - f_{uv}^2}{W^4}.$$

$$H = \frac{Gl + En - 2Fm}{2(EG - F^2)} = \frac{((1 + f_v^2) f_{uu} + (1 + f_u^2) f_{vv} - 2 f_u f_v f_{uv})/W}{2W^2} = \frac{(1 + f_v^2) f_{uu} + (1 + f_u^2) f_{vv} - 2 f_u f_v f_{uv}}{2W^3}.$$

Solution 4.2

(a) From the previous solution, the surface is flat if, and only if, $K = 0$. This means that the requirement is

$$f_{uu} f_{vv} - f_{uv}^2 = 0.$$

(b) The result follows directly by requiring the numerator of the expression for H to be zero.

Solution 4.3

You may have worked from first principles or used the results of the first exercise. We do the latter.

Here

$$f(u, v) = \log \cos v - \log \cos u$$
$$f_u(u, v) = \frac{\sin u}{\cos u} = \tan u;$$
$$f_v(u, v) = -\frac{\sin v}{\cos v} = -\tan v;$$
$$W^2 = (1 + f_u^2 + f_v^2) = 1 + \tan^2 u + \tan^2 v.$$

Hence

$$f_{uu}(u, v) = \sec^2 u,$$
$$f_{uv}(u, v) = 0,$$
$$f_{vv}(u, v) = -\sec^2 v.$$

Substituting in the formulas

$$K = \frac{f_{uu} f_{vv} - f_{uv}^2}{W^4} = \frac{-\sec^2 u \sec^2 v}{W^4},$$

$$H = \frac{(1 + f_v^2) f_{uu} + (1 + f_u^2) f_{vv} - 2 f_u f_v f_{uv}}{2W^3} = \frac{(1 + \tan^2 v) \sec^2 u - (1 + \tan^2 u) \sec^2 v}{2W^3} = \frac{\sec^2 v \sec^2 u - \sec^2 u \sec^2 v}{2W^3} = 0.$$

Hence M is minimal and K is as stated.

Solution 5.1

Being an 'if and only if' proof, this solution is in two parts.

First, assume that α is a straight line in $\mathbf{E}^3$. Then

$$\alpha'' = 0.$$

Since $\alpha'' \cdot U = 0$, the acceleration of α is (trivially) normal to the surface M and so α is a geodesic.

Also

$$\alpha'' = 0 \times \mathbf{x}_u + 0 \times \mathbf{x}_v$$

for any patch of M and so the acceleration is also tangent to M. Thus α is also asymptotic.

Conversely, suppose that α is geodesic and asymptotic. Then α'' is simultaneously tangent to and normal to M. Hence

$$\alpha'' = 0$$

and α is a straight line in $\mathbf{E}^3$.

Solution 5.2

An algebraic approach uses the information assembled, in *O'Neill*'s examples about the torus:

$$\mathbf{x}_u = (-r\sin u\cos v, -r\sin u\sin v, r\cos u),$$
$$\mathbf{x}_v = (-(R + r\cos u)\sin v, (R + r\cos u)\cos v, 0),$$
$$U = (\cos u\cos v, \cos u\sin v, \sin u) \quad \text{(outward normal)}.$$

(a) This curve is the parameter curve defined by $u = \pi/2$. That is

$$\begin{aligned}\alpha(t) &= \mathbf{x}(\pi/2, t)\\ &= (R\cos t, R\sin t, r).\end{aligned}$$

Thus, on the curve, we have that the velocity is $\mathbf{x}_v(\pi/2, t)$ and so

$$\begin{aligned}\alpha'(t) &= (-R\sin t, R\cos t, 0),\\ U(\alpha(t)) &= (0, 0, 1).\end{aligned}$$

Hence

$$\begin{aligned}\alpha''(t) &= (-R\cos t, -R\sin t, 0),\\ U' &= (0, 0, 0).\end{aligned}$$

Using the various tests, we have

$$U' = 0\alpha',$$
$$U \cdot \alpha'' = 0,$$
$$U \text{ is orthogonal to } \alpha''.$$

Thus α is principal, from the first test, and the principal curvature is zero along the curve.

From the second test we deduce that α is asymptotic. (This fits with the result that it is principal with principal curvature zero.)

Finally, the last observation shows that α is not geodesic.

Note: Geometrically, these results accord with intuition. The unit normal on the curve points upwards from the plane of the curve (which is a circle). The acceleration points towards the centre of the circle and is, therefore, tangent to the surface. Finally, the unit normal is parallel on the curve and so the normal curvature is zero along it. (This last sentence meaning that the normal curvature in the direction of the tangent to the curve is zero.)

(b) This curve is the parameter curve $u = 0$. Thus, as above, we have

$$\begin{aligned}\beta(t) &= \mathbf{x}(0, t),\\ \beta'(t) &= (-(R + r)\sin t, (R + r)\cos t, r),\\ U(\beta(t)) &= (\cos t, \sin t, 0).\end{aligned}$$

Hence

$$\begin{aligned}\beta''(t) &= (-(R + r)\cos t, -(R + r)\sin t, 0),\\ U' &= (-\sin t, \cos t, 0).\end{aligned}$$

Using the various tests, we have

$$U' = \frac{\beta'}{(R + r)},$$
$$U \cdot \beta'' = -(R + r) \neq 0,$$
$$U = -\frac{\beta''}{(R + r)}.$$

Hence β is principal, with principal curvature

$$\begin{aligned}\frac{\beta'' \cdot U}{\beta' \cdot \beta'} &= \frac{-(R + r)}{(R + r)^2}\\ &= -\frac{1}{(R + r)}.\end{aligned}$$

The curve is not asymptotic, but is geodesic.

(c) We may apply the same sort of techniques as above and in the example in the text. Since

$$\mathbf{x}(u, v) = (u, v, uv),$$

the x-axis can be described as the parameter curve $v = 0$, giving

$$\alpha(t) = (t, 0, 0).$$

Since

$$\alpha'(t) = (1, 0, 0),$$
$$\alpha''(t) = (0, 0, 0),$$

we can see that the acceleration is (trivially) both tangent to and normal to the surface.

Along α, we have

$$U = \frac{(0, -t, 1)}{\sqrt{1 + t^2}},$$

and so

$$U' = \frac{(0, -1, -t)}{(1 + t^2)^{3/2}}.$$

Since $U' \cdot \alpha' = 0$, U' and α' are orthogonal and so the curve is asymptotic. Also, since U' and α' are not collinear, so α is not principal. Since the acceleration is (trivially) a multiple of the unit normal, the curve is geodesic.

Solution 5.3

We assemble a few preliminary facts. From the Frenet formulas, we have that $\alpha' = T$ and so

$$\alpha'' = T' = \kappa N.$$

(a) Since α is asymptotic, we know that its acceleration is tangent to M. Hence, from the remarks above, N is tangent to M.

But, since α is a curve in M, its velocity T is also tangent to M. Hence

$$B = T \times N$$

is normal to M.

(b) Applying the result that, along a curve, the shape operator is given by $-U'$, we have

$$S(T) = -B' = -(-\tau N) = \tau N,$$

by using the Frenet formula for B'.

Solution 6.1

We can apply the general discussion of surfaces of revolution in *O'Neill* to this case.

If we set

$$g(u) = a\cos u,$$
$$h(u) = b\sin u,$$

then we obtain the parametrization

$$\mathbf{x}(u, v) = (a\cos u, b\sin u\cos v, b\sin u\sin v) \quad 0 < u < \pi$$

of M.

(a) We have

$$\begin{aligned} g'(u) &= -a\sin u, \\ g''(u) &= -a\cos u, \\ h'(u) &= b\cos u, \\ h''(u) &= -b\sin u, \\ g'^2 + h'^2 &= a^2\sin^2 u + b^2\cos^2 u, \\ g'h'' - g''h' &= ab(\sin^2 u + \cos^2 u) \\ &= ab. \end{aligned}$$

Hence

$$\begin{aligned} K &= \frac{-g'(g'h'' - g''h')}{h(g'^2 + h'^2)^2} \\ &= \frac{a^2 b\sin u}{b\sin u(a^2\sin^2 u + b^2\cos^2 u)^2} \\ &= \frac{a^2}{(a^2\sin^2 u + b^2\cos^2 u)^2}. \end{aligned}$$

(b) From the formula for K above, we can see that K is a maximum when the denominator is a minimum and vice versa. It is slightly easier to find the extreme values of the denominator if we express it in terms of $\cos u$ only.

Since

$$\begin{aligned} a^2\cos^2 u + b^2\sin^2 u &= a^2\cos^2 u + b^2(1 - \cos^2 u) \\ &= b^2 + (a^2 - b^2)\cos^2 u, \end{aligned}$$

the maximum value of the denominator occurs when $u = 0$ or $u = \pi$ and the minimum when $u = \pi/2$.

Thus the maximum Gaussian curvature is

$$K(\mathbf{x}(0, v)) = K(\mathbf{x}(\pi, v)) = \frac{a^2}{b^4},$$

and occurs for all points with $u = 0$, i.e. at $(\pm a, 0, 0)$.

The minimum Gaussian curvature is

$$K(\mathbf{x}(\pi/2, v)) = \frac{a^2}{a^4} = \frac{1}{a^2}$$

and occurs for all points on the meridian $u = \pi/2$.

Solution 6.2

If we assume that the surface has been parametrized in such a way that the profile curve is unit speed, that is using the 'canonical' parametrization, we have

$$K = \frac{-h''}{h}.$$

We must, therefore, solve $K = 0$, that is

$$-h'' = 0,$$

subject to the conditions $h > 0$ and $\|h'\| < 1$.

The general solution of $h'' = 0$ is

$$h(t) = at + b,$$

where a and b are constants.

Since $h'(t) = a$, the condition $\|h'\| < 1$ requires

$$\|a\| < 1.$$

Next, we apply the formula from page 239 to obtain g.

$$\begin{aligned} g(u) &= \int_0^u \sqrt{1 - (h'(t))^2}\,dt \\ &= \int_0^u \sqrt{1 - a^2}\,dt \\ &= \left[\sqrt{(1 - a^2)}\,t\right]_0^u \\ &= u\sqrt{1 - a^2}. \end{aligned}$$

Thus, the profile curve is

$$\alpha(t) = (u\sqrt{1 - a^2}, au + b).$$

We can reparametrize this by replacing u by

$$\frac{u}{\sqrt{1 - a^2}}$$

to get a profile curve of the form

$$(u, cu + d),$$

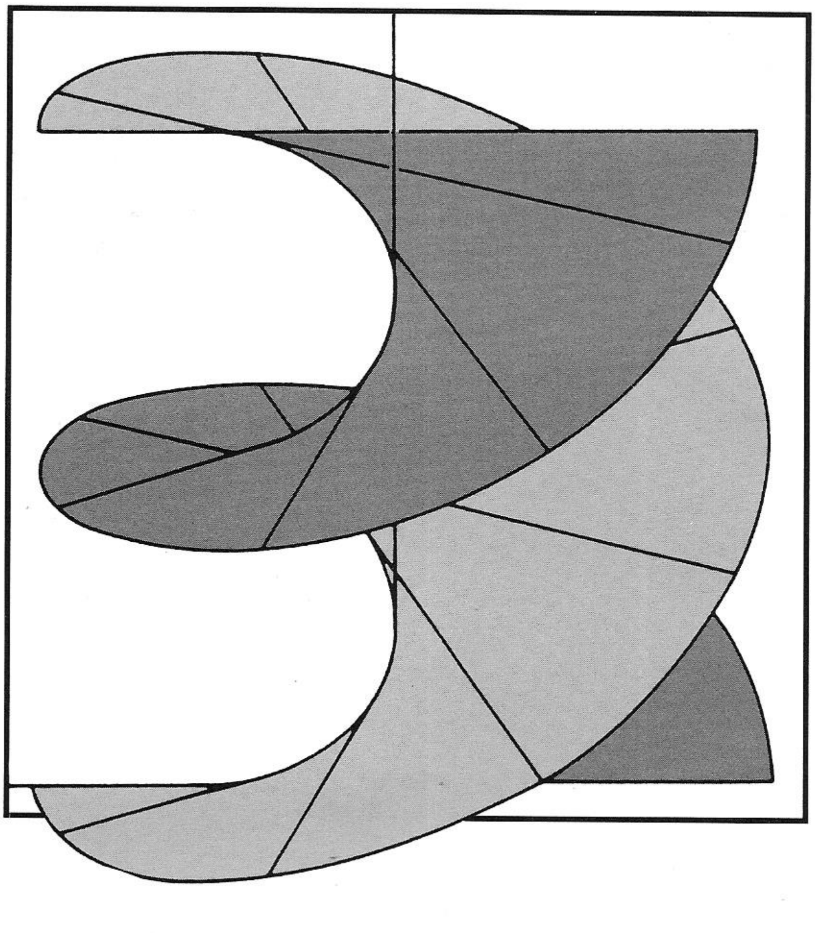
where c and d are constants.

Thus the profile curves are straight lines and the surfaces are parts of cones (if the slope c is not zero) or parts of cylinders (if $c = 0$).

ISBN 0 7492 4774 6

M434 Part VI
Mathematics: A Fourth Level Course

M434
DIFFERENTIAL GEOMETRY



PART VI
GEOMETRY OF SURFACES

The Open University

The Open University

Mathematics: A Fourth Level Course

M434 Differential Geometry

Part VI Geometry of Surfaces

Prepared for the Course Team
by Bob Margolis

Set book

Barrett O'Neill, *Elementary Differential Geometry*, hardback edition (Academic Press, 1966).

It is essential to have this book; the course is based on it and will not make sense without it.

The set book is referred to as *O'Neill*.

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published as *M334* 1976.

First published as *M434* 1993. Reprinted 1996.

Copyright © 1993 The Open University

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, without written permission from the publisher or a licence from the Copyright Licensing Agency Limited. Details of such licences (for reprographic reproduction) may be obtained from the Copyright Licensing Agency Ltd of 90 Tottenham Court Road, London, W1P 9HE.

Edited, designed and typeset by the Open University using the Open University TEX System.

Printed in the United Kingdom by Hobbs the Printers Ltd, Totton, Hampshire.

ISBN 0 7492 4775 4

This text forms part of an Open University Fourth Level Course. If you would like a copy of *Studying with The Open University*, please write to the Central Enquiry Service, PO Box 200, The Open University, Walton Hall, Milton Keynes, MK7 6YZ. If you have not already enrolled on the Course and would like to buy this or other Open University material, please write to Open University Educational Enterprises Ltd, 12 Cofferidge Close, Stony Stratford, Milton Keynes, MK11 1BY, United Kingdom.

1.2

Contents

Introduction 4

1 The fundamental equations 4

2 Form computations 6

3 Orthogonal coordinates 7

4 The Gauss map 9

5 Summary 11

Solutions to the exercises 12

Introduction

Our study of surfaces has not made direct use of differential forms, nor of the connection and structural equations which we took so much trouble to develop earlier. We have pointed out the links with the equations from time to time, now we summarize these links and pursue the 'form' approach a little further.

There are two main themes. Firstly, we shall show how differential forms on a surface can be used to calculate Gaussian curvature and, hence, how they may be used to *define* Gaussian curvature for surfaces in general. Secondly, we shall return to the Gauss map (from a surface to the unit sphere) and link it to area 2-forms and Gaussian curvature.

The first two sections are based on reading *O'Neill*, Sections 1 and 2. They form the beginnings of *O'Neill*'s program of investigating surfaces from the point of view of inhabitants of the surface who know nothing about $\mathbf{E}^3$.

Section 3 is based on Section 6 of *O'Neill*. This section does not depend, in any essential way, on the three omitted sections of *O'Neill*. It contains useful, practical methods for calculating Gaussian curvature.

Section 4 does not have a reading passage from *O'Neill*. It pulls together the Gauss map, area 2-forms and the 'pullback' of the Gauss map.

The pullback of the mapping F is F^*, as defined in *Part IV*, Section 5.

Section 5 is a brief summary.

Study advice

The final TMA for the course has questions on Sections 4–6 of *Part V* and all of *Part VI*. The following represents a possible plan for study weeks.

Week 1 *O'Neill*, Chapter V, Sections 4 and 5.

Week 2 *O'Neill*, Chapter V, Section 6 and Chapter VI, Section 1.

Week 3 *O'Neill*, Chapter VI, Sections 2 and 3.

Week 4 Sections 4 and 5 of this text and TMA04.

1 The fundamental equations

Read *O'Neill, Chapter VI, Section 1, pages 245–250.*

This section draws together a number of the remarks that we have made about the links between the shape operator, defined in terms of a unit normal vector field, and the connection forms of *Part II*.

Throughout this commentary, we shall assume that E_1, E_2 and E_3 is an adapted frame field on at least part of a surface M.

Dual 1-forms The dual 1-forms for a general frame field are completely determined by the equations

$$\theta_i(E_j) = \delta_{ij}, \quad i, j = 1, 2, 3.$$

The remark on page 248, that $\theta_3 = 0$, is a little peculiar, for a tangent vector may be attached to a surface in $\mathbf{E}^3$ without, necessarily, being tangent to the surface. Such a vector may well have a non-zero E_3-component.

However, the purpose of this section is to begin disposing of all the concepts that belong to the surface because it lies in $\mathbf{E}^3$. Tangents to the surface will be retained, normals, including E_3 will not. Thus θ_3 disappears along with E_3!

Remark Once we have the dual 1-forms, all 1-forms on the surface can be expressed as linear combinations of θ_1 and θ_2, including the connection forms. We shall use this fact in Section 3.

Similarly, all 2-forms on the surface can be expressed in terms of the wedge product $\theta_1 \wedge \theta_2$.

Theorem 1.7 As with many such results in *O'Neill*, the proof consists of writing out the connection and structural equations in full, extracting the particular components required and then substituting any special information. Here the last step is to use $\theta_3 = 0$.

As an example and reminder, we have

$$d\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix} = \begin{pmatrix}0 & \omega_{12} & \omega_{13}\\ -\omega_{12} & 0 & \omega_{23}\\ -\omega_{13} & -\omega_{23} & 0\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix}.$$

Matrix multiplication and substitution of

$$\theta_3 = d\theta_3 = 0$$

give precisely the first three results of the theorem.

Exercise 1.1 Suppose that E_1, E_2 and E_3 form an adapted frame field on the surface $M \subset \mathbf{E}^3$. Suppose also that α is a curve in M such that the velocity of α is the restriction of E_1 to α. That is,

$$E_1(\alpha(t)) = \alpha'(t).$$

(a) Explain why, with the usual notation, $\nabla_{E_1} V = V'$, for any vector field V on α.

(b) Show that α is a geodesic if, and only if, $\omega_{12}(E_1) = 0$. (*Hint*: Apply the definition of *geodesic* and the result of part (a).)

Exercise 1.2 This question uses the same notation as the previous exercise. Show that the normal curvature along α is $\omega_{13}(E_1)$.

Exercise 1.3 Let T be the torus parametrized by

$$\mathbf{x}(u,v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u), \quad R > r > 0.$$

This question indulges in the usual abuse of notation in regarding u and v as being defined on T rather than on the domain of $\mathbf{x}$.

(a) Show that the partial velocities can be used to define orthogonal unit tangent vector fields E_1, E_2 as follows:

$$E_1 = \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|}, \quad E_2 = \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}.$$

Define the third member E_3 of an adapted frame field on T.

(b) Calculate

$$du(E_1), \quad du(E_2), \quad dv(E_1) \quad \text{and} \quad dv(E_2).$$

Hence show that the dual 1-forms are given by

$$\theta_1 = r\,du, \quad \theta_2 = (R + r\cos u)dv.$$

(c) By direct calculation show that

$$d\theta_1 = 0, \quad d\theta_2 = -r\sin u\,du\,dv$$

and deduce that

$$d\theta_2 = -\frac{\sin u}{R + r\cos u}\theta_1 \wedge \theta_2.$$

(d) Since ω_{12} is a 1-form on T, it is a linear combination

$$\omega_{12} = a\theta_1 + b\theta_2$$

of the dual 1-forms. By substituting into the first structural equations (Lemma 1.7(1)), show that

$$\omega_{12} = -\frac{\sin u}{R + r\cos u}\theta_2.$$

(e) Show that

$$d\omega_{12} = -\frac{\cos u}{r(R + r\cos u)}\theta_1 \wedge \theta_2.$$

(*Hint*: When differentiating the expression for ω_{12} you will need the Leibniz property and the value of $d\theta_2$ already found.)

[*Solutions on page 12*]

2 Form computations

Read *O'Neill, Chapter VI, Section 2, pages 251–255.*

Erratum *O'Neill*, page 252, the calculation of $S(E_1)$, $S(E_2)$ and the two lines following should read:

$$\begin{aligned} S(E_1) &= -\nabla_{E_1} E_1 \\ &= -(\omega_{31}(E_1)\, E_1 + \omega_{32}(E_1)\, E_2) \\ &= \omega_{13}(E_1)\, E_1 + \omega_{23}(E_1)\, E_2 \quad \text{(alternation)} \end{aligned}$$

$$\begin{aligned} S(E_2) &= -\nabla_{E_2} E_1 \\ &= -(\omega_{31}(E_2)\, E_1 + \omega_{32}(E_2)\, E_2) \\ &= \omega_{13}(E_2)\, E_1 + \omega_{23}(E_2)\, E_2 \quad \text{(alternation).} \end{aligned}$$

Thus the matrix of S with respect to $\{E_1, E_2\}$ is

$$\begin{pmatrix} \omega_{13}(E_1) & \omega_{13}(E_2) \\ \omega_{23}(E_1) & \omega_{23}(E_2) \end{pmatrix}.$$

■

This section gathers together a number of the ideas on which we have already remarked in the commentaries to earlier chapters of *O'Neill*.

In *Part V*, Section 3 we obtained the matrix on page 252 and remarked that

$$K = (\omega_{13} \wedge \omega_{23})(E_1, E_2) = -d\omega_{12}(E_1, E_2).$$

The observations in Lemma 2.1 immediately give Corollary 2.3.

It is important to realize that the significance of Corollary 2.3 is not so much that it is yet another formula for calculating Gaussian curvature but that it contains only forms that belong to the geometry of the surface. None of the items in the formula

$$d\omega_{12} = -K\theta_1 \wedge \theta_2$$

are dependent on the surface being in $\mathbf{E}^3$. It follows that, if we want to generalize our work to abstract surfaces, Corollary 2.3 can become the *definition* of Gaussian curvature.

This is not an unfamiliar process. We have already used a lemma:

$$\alpha'[f] = (f(\alpha))',$$

about directional derivatives of functions with respect to the velocity of a curve in $\mathbf{E}^3$, to formulate the definition of directional derivative for functions defined on surfaces.

These exercises continue the investigation of the torus begun in the second exercise of the last section. You should use the results obtained there.

Exercise 2.1 Write down the Gaussian curvature of the torus T.

Exercise 2.2 Is the frame field (in Exercise 2.1) principal?

[*Solutions on page 13*]

3 Orthogonal coordinates

Read *O'Neill, Chapter VI, Section 6, pages 276–279.*

This section generalizes the work you have done (in various exercises) on the torus.

The feature that was responsible for simplifying many of the calculations on the torus was the orthogonality of $\mathbf{x}_u$ and $\mathbf{x}_v$. The fact that the parameter curves were principal was merely a bonus.

If you inspect the calculations in this section carefully and compare them with the solutions to the torus exercises, you should see the analogies.

Note how, at the top of page 277, *O'Neill* finally agrees that the abuse of notation that we have been committing is reasonable.

Finding ω_{12} If you cannot see how *O'Neill* did the 'comparison with the first structural equations' on page 277, we suggest the following systematic approach.

Since ω_{12} is a 1-form, it can be written

$$\omega_{12} = a\,du + b\,dv,$$

for some functions a and b on M.

Thus

$$\begin{aligned} -\frac{(\sqrt{E})_u}{\sqrt{G}}\,du \wedge \theta_2 = d\theta_1 & \\ &= \omega_{12} \wedge \theta_2 \\ &= (a\,du + b\,dv) \wedge \sqrt{G}dv \\ &= a\sqrt{G}\,du\,dv \\ &= a\,du \wedge \theta_2. \end{aligned}$$

It follows that

$$a = -\frac{(\sqrt{E})_u}{\sqrt{G}}.$$

The value of b is obtained from

$$d\theta_2 = -\omega_{12} \wedge \theta_1$$

in exactly the same way.

Lemma 6.3 We think that *O'Neill* gives very sound advice. It is the derivation of this lemma that is important. For any particular example it is better to work directly from the structural equations than to memorize the result of Lemma 6.3.

When following this advice there is a choice to be made: whether to work in terms of du and dv or the dual 1-forms θ_1 and θ_2.

The calculation of the exterior derivatives is slightly easier in terms of du and dv because of the simple formulas

$$\begin{aligned} df &= f_u\,du + f_v\,dv \\ d(df) &= 0. \end{aligned}$$

It is theoretically more 'elegant' to work in terms of the dual 1-forms.

The main difference comes when calculating $d\omega_{12}$. Suppose that

$$\omega_{12} = a\,du + b\,dv = f\theta_1 + g\theta_2.$$

Then

$$\begin{aligned} d\omega_{12} &= (da\,du + a\,d(du)) + (db\,dv + b\,d(dv)) \\ &= (a_u\,du + a_v\,dv)du + (b_u\,du + b_v\,dv)dv \\ &= a_v\,dv\,du + b_u\,du\,dv \\ &= (b_u - a_v)du\,dv, \end{aligned}$$

Using the other expansion gives

$$\begin{aligned} d\omega_{12} &= (df \wedge \theta_1 + f\,d\theta_1) + (dg \wedge \theta_2 + g\,d\theta_2) \\ &= df \wedge \theta_1 + dg \wedge \theta_2 + f\,d\theta_1 + g\,d\theta_2 \\ &= df \wedge \theta_1 + dg \wedge \theta_2 + f\,\omega_{12} \wedge \theta_2 - g\,\omega_{12} \wedge \theta_1 \\ &= df \wedge \theta_1 + dg \wedge \theta_2 + \omega_{12} \wedge (f\theta_2 - g\theta_1) \\ &= df \wedge \theta_1 + dg \wedge \theta_2 + (f\theta_1 + g\theta_2) \wedge (f\theta_2 - g\theta_1) \\ &= df \wedge \theta_1 + dg \wedge \theta_2 + (f^2 - g^2)\theta_1 \wedge \theta_2. \end{aligned}$$

There is a return for taking the extra trouble to work in terms of the dual 1-forms. Since df and dg are 1-forms, they can be expressed in terms of θ_1 and θ_2. Thus all terms in the expression for $d\omega_{12}$ automatically appear in terms of $\theta_1 \wedge \theta_2$ which enables you to read off the Gaussian curvature from

$$d\omega_{12} = -K\theta_1 \wedge \theta_2.$$

We continue the investigation of the torus with the usual parametrization.

Exercise 3.1 Calculate E, F and G for the standard parametrization of the torus and hence calculate K from the formula in Lemma 6.3.

Exercise 3.2 This exercise is an alternative approach to Exercise 3.1.

(a) Using the formulas on page 277, write down expressions for the dual 1-forms in terms of du and dv.

(b) Calculate $d\theta_1$ and $d\theta_2$ in terms of $du\,dv$.

(c) Assume that $\omega_{12} = a\,du + b\,dv$. and use the first structural equation to find a and b.

(d) Calculate $d\omega_{12}$ in terms of $du\,dv$.

(e) Use the second structural equation to calculate the Gaussian curvature K.

[Solutions on page 13]

4 The Gauss map

There is no reading from *O'Neill* for this section. We now take up the second theme of this part: the link between the Gauss map, area 2-forms and Gaussian curvature in the rather special case of a compact orientable surface.

Chapter VII of *O'Neill* does deal with more general considerations.

We assume the following situation. We have a surface M parametrized by a single mapping $\mathbf{x}(u,v)$ and the unit sphere Σ. We shall define a parametrization of the sphere shortly.

We assume that M has a unit normal vector field U obtained from

$$U(\mathbf{x}(u,v)) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}.$$

The Gauss map F from M to Σ is defined by

$$F(\mathbf{x}(u,v)) = U(\mathbf{x}(u,v)),$$

where we interpret $U(\mathbf{x}(u,v))$ as giving the coordinates of a point on the unit sphere. Since $U(\mathbf{x}(u,v))$ is unit length, its coordinate functions do define a point on the unit sphere.

Now we define a parametrization of the sphere by

$$\mathbf{y}(u,v) = U(\mathbf{x}(u,v)) = F(\mathbf{x}(u,v)).$$

The sphere Σ also has a unit normal vector field V. Because it is a unit sphere, we can see that V has the same coordinate functions as $\mathbf{y}$:

$$V(\mathbf{y}(u,v)) = \mathbf{y}(u,v) = U(\mathbf{x}(u,v)) = F(\mathbf{x}(u,v)).$$

If we ignore points of application, $U = V$; more precisely, U and V are parallel for all values of u and v.

The Gauss map has a number of interesting properties. We consider two of them. Firstly, because, by definition

$$\mathbf{y}(u,v) = F(\mathbf{x}(u,v)),$$

we have

$$\mathbf{y}_u(u,v) = F_*(\mathbf{x}_u(u,v)),$$
$$\mathbf{y}_v(u,v) = F_*(\mathbf{x}_v(u,v)).$$

Here the special case of the effect of derived maps on page 161 of *O'Neill is* applicable.

Secondly, if we ignore points of application in a thoroughly cavalier fashion,

$$\begin{aligned} F_*(\mathbf{x}_u(u,v)) &= \frac{\partial}{\partial u} F(\mathbf{x}(u,v)) \\ &= \frac{\partial}{\partial u} U(\mathbf{x}(u,v)) \\ &= U_u(\mathbf{x}(u,v)) \\ &= -S(\mathbf{x}_u(u,v)). \end{aligned}$$

More precisely, we could say that $F_*(\mathbf{x}_u(u,v))$ and $-S(\mathbf{x}_u(u,v))$ are parallel for all u and v. Thus $\mathbf{y}_u(u,v)$ is parallel to $-S(\mathbf{x}_u(u,v))$ and similarly for the other partial velocity.

Next we define area 2-forms on both surfaces. We need only define their effect on the partial velocities as the partial velocities form a basis for the tangent plane at each point.

On M we define the 2-form η by

η pronounced 'eta'.

$$\eta(\mathbf{x}_u, \mathbf{x}_v) = \mathbf{x}_u \times \mathbf{x}_v \cdot U$$

and on Σ we define ζ by

ζ pronounced 'zeta'.

$$\zeta(\mathbf{y}_u, \mathbf{y}_v) = \mathbf{y}_u \times \mathbf{y}_v \cdot V.$$

We now have two 2-forms on M, the area 2-form η defined above and the pullback of ζ from Σ to M via the Gauss map F. This second 2-form is defined by

Part IV, Section 5.

$$(F^*\zeta)(\mathbf{x}_u, \mathbf{x}_v) = \zeta(F_*(\mathbf{x}_u), F_*(\mathbf{x}_v)).$$

We now explore the relation between $F^*\zeta$ and η.

We can calculate the effect of η and ζ on the appropriate partial velocities quite easily.

$$\begin{aligned}\eta(\mathbf{x}_u, \mathbf{x}_v) &= \mathbf{x}_u \times \mathbf{x}_v \cdot U \\ &= \|\mathbf{x}_u \times \mathbf{x}_v\| U \cdot U \quad \text{(definition of } U\text{)} \\ &= \|\mathbf{x}_u \times \mathbf{x}_v\|.\end{aligned}$$

Suppose that

$$\begin{aligned}S(\mathbf{x}_u) &= a\mathbf{x}_u + b\mathbf{x}_v, \\ S(\mathbf{x}_v) &= c\mathbf{x}_u + d\mathbf{x}_v,\end{aligned}$$

for suitable functions a, b, c, d.

Then

$$\begin{aligned}\zeta(\mathbf{y}_u, \mathbf{y}_v) &= \mathbf{y}_u \times \mathbf{y}_v \cdot V \\ &= (-S(\mathbf{x}_u)) \times (-S(\mathbf{x}_v)) \cdot U \\ &= (a\mathbf{x}_u + b\mathbf{x}_v) \times (c\mathbf{x}_u + d\mathbf{x}_v) \cdot U \\ &= (ad - bc)\mathbf{x}_u \times \mathbf{x}_v \cdot U \\ &= (ad - bc)\|\mathbf{x}_u \times \mathbf{x}_v\|.\end{aligned}$$

But $ad - bc$ is just the determinant of the shape operator S, that is the Gaussian curvature. Hence

$$\zeta(\mathbf{y}_u, \mathbf{y}_v) = K\eta(\mathbf{x}_u, \mathbf{x}_v).$$

Now consider the pullback $F^*\zeta$.

$$\begin{aligned}F^*\zeta(\mathbf{x}_u, \mathbf{x}_v) &= \zeta(F_*(\mathbf{x}_u), F_*(\mathbf{x}_v)) \\ &= \zeta(\mathbf{y}_u, \mathbf{y}_v) \\ &= K\eta(\mathbf{x}_u, \mathbf{x}_v).\end{aligned}$$

We have shown that $F^*\zeta$ and $K\eta$ agree on the linearly independent pair $(\mathbf{x}_u, \mathbf{x}_v)$ at each point so they are equal as 2-forms. That is

$$F^*\zeta = K\eta.$$

This provides yet another approach to finding Gaussian curvature. However, it is only in some simple cases that such an approach is less trouble than previous ones.

Perhaps it is of more interest because it generalizes the observation made earlier about what happens to areas under mappings. In the simple case of a matrix mapping from the plane to itself, it is the determinant of the matrix that controls how areas are mapped. In the mapping of a surface to a sphere, it is the determinant of the shape operator that controls mapping of area 2-forms.

Remark: You may have worried that there are two choices for the normal on the surface and also on the sphere, hence two area 2-forms on each surface. You may care to check that, provided we always define the Gauss map by using the chosen normal on M, and hence define $V = U$, the result is exactly the same.

We set one, simple, application of the above.

Exercise 4.1 Let T be the torus parametrized by

$$\mathbf{x}(u, v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u), \quad R > r > 0$$

and Σ be the unit sphere parametrized by

$$\mathbf{y}(u, v) = (\cos u \cos v, \cos u \sin v, \sin u).$$

Let η and ζ be their respective area 2-forms.

(a) Use the partial velocities to define an outward unit normal vector field U on T.

(b) Show that, with the above choice of U,

$$F : \mathbf{x}(u,v) \longmapsto \mathbf{y}(u,v)$$

is the Gauss map of the torus to the sphere.

(c) Calculate directly

$$\eta(\mathbf{x}_u(u,v), \mathbf{x}_v(u,v))$$

and

$$\zeta(\mathbf{y}_u(u,v), \mathbf{y}_v(u,v)).$$

(d) Calculate

$$F^*\zeta(\mathbf{x}_u(u,v), \mathbf{x}_v(u,v)).$$

Hence deduce the Gaussian curvature of the torus.

[*Solution on page 14*]

5 Summary

This text has mainly been concerned with tying together some of the threads that have run through the course. We have shown how the shape operator ties in with the connection forms and indicated how this link might be used to define Gaussian curvature in more general work.

There is more in *O'Neill* than is appropriate to include in a half-credit course. We have tried to ensure that the material included is coherent, whilst still providing you with a sound basis for further reading.

Underlying all the theorems and computational techniques are the ideas from Chapter I, particularly the linearity and Leibniz properties of the various types of derivative.

You may well have found that the amount of new notation in the early chapters was somewhat daunting; by now the reasons for introducing it all should be clearer. Differential forms, in particular, provide a notationally cleaner approach than the use of tensors that you may find in earlier textbooks. The difference is similar to using matrices, rather than always working with coordinates, in linear algebra.

You should now be able to calculate the 'curvature apparatus' for curves and surfaces with some confidence. Although there appear to be quite a large number of techniques, they are all based upon use of a small collection of principles.

Solutions to the exercises

Solution 1.1

(a) By the definition of covariant derivative on a surface and the hypotheses about α, we have

$$\begin{aligned}\nabla_{E_1} V &= \nabla_{\alpha'} V\\ &= (V(\alpha))'\\ &= V'.\end{aligned}$$

The last step is simply what is meant by V'.

(b) By definition, α is a geodesic if, and only if, its acceleration is normal to M. We must, therefore, calculate $\alpha'' = E_1'$.

Applying the above result, we have

$$E_1' = \nabla_{E_1} E_1.$$

Now we can apply the connection equations to get

$$\begin{aligned}E_1' &= \nabla_{E_1} E_1\\ &= \omega_{12}(E_1)\, E_2 + \omega_{13}(E_1)\, E_3.\end{aligned}$$

Since E_3 is normal to M, α is geodesic precisely when the E_2 term in the acceleration is zero, that is, when

$$\omega_{12}(E_1) = 0.$$

Solution 1.2

The required normal curvature is

$$S(E_1) \cdot E_1 = -\nabla_{E_1} E_3 \cdot E_1.$$

From the connection equations,

$$\begin{aligned}\nabla_{E_1} E_3 &= \omega_{31}(E_1) E_1 + \omega_{32}(E_1) E_2\\ &= -\omega_{13}(E_1) E_1 - \omega_{23}(E_1) E_2.\end{aligned}$$

Hence

$$\begin{aligned}S(E_1) \cdot E_1 &= -(-(\omega_{13}(E_1) E_1 - \omega_{23}(E_1) E_2)) \cdot E_1\\ &= \omega_{13}(E_1),\end{aligned}$$

as required.

Note: You could have shortened the working slightly by applying Corollary 1.5 directly.

Solution 1.3

(a) First, we calculate the partial velocities.

$$\begin{aligned}\mathbf{x}_u &= (-r \sin u \cos v, -r \sin u \sin v, r \cos u),\\ \mathbf{x}_v &= (-(R + r\cos u) \sin v, (R + r \cos u) \cos v, 0).\end{aligned}$$

We also have

$$\|\mathbf{x}_u\| = r,\ \|\mathbf{x}_v\| = (R + r \cos u).$$

Thus

$$\begin{aligned}E_1 &= (-\sin u \cos v, -\sin u \sin v, \cos u),\\ E_2 &= (-\sin v, \cos v, 0)\end{aligned}$$

are certainly unit length. It is also clear that

$$E_1 \cdot E_2 = 0,$$

so they are orthogonal. The third member can now be defined by

$$\begin{aligned}E_3 &= E_1 \times E_2\\ &= (-\cos u \cos v, -\cos u \sin v, -\sin u).\end{aligned}$$

(b) Since du and dv pick out the $\mathbf{x}_u$ and $\mathbf{x}_v$ components respectively of a vector tangent to T, we have

$$\begin{aligned}du(E_1) &= du\left(\frac{\mathbf{x}_u}{\|\mathbf{x}_u\|}\right)\\ &= \frac{1}{r},\\ du(E_2) &= du\left(\frac{1}{R + r\cos u}\mathbf{x}_v\right)\\ &= 0,\\ dv(E_1) &= dv\left(\frac{1}{r}\mathbf{x}_u\right)\\ &= 0,\\ dv(E_2) &= dv\left(\frac{1}{R + r\cos u}\mathbf{x}_v\right)\\ &= \frac{1}{R + r\cos u}.\end{aligned}$$

Comparing these with

$$\theta_i(E_j) = \delta_{ij},$$

we can see that

$$\begin{aligned}du &= \frac{1}{r}\theta_1,\\ dv &= \frac{1}{R + r\cos u}\theta_2.\end{aligned}$$

The required results follow immediately.

(c) Differentiating

$$\begin{aligned}d\theta_1 &= dr\, du\\ &= 0 du\\ &= 0,\\ d\theta_2 &= d(R + r\cos u)\, dv\\ &= -r \sin u\, du\, dv\\ &= -r \sin u \frac{\theta_1}{r} \wedge \frac{\theta_2}{R + r\cos u}\\ &= -\frac{\sin u}{R + r\cos u}\theta_1 \wedge \theta_2.\end{aligned}$$

Substituting into

$$d\theta_1 = \omega_{12} \wedge \theta_2,$$

we have

$$\begin{aligned}0 &= (a\theta_1 + b\theta_2) \wedge \theta_2\\ &= a\theta_1 \wedge \theta_2 + 0\\ &= a\theta_1 \wedge \theta_2.\end{aligned}$$

Hence $a = 0$ and $\omega_{12} = b\theta_2$.

Using the second equation

$$\begin{aligned}d\theta_2 &= -\frac{\sin u}{R + r\cos u}\theta_1 \wedge \theta_2\\ &= -b\theta_2 \wedge \theta_1\\ &= b\theta_1 \wedge \theta_2.\end{aligned}$$

Hence

$$b = -\frac{\sin u}{R + r\cos u}$$

and

$$\omega_{12} = -\frac{\sin u}{R + r\cos u}\theta_2.$$

(d) Using the hint, we have

$$\begin{aligned}
d\omega_{12} &= d\left(-\frac{\sin u}{R+r\cos u}\right)\theta_2 + \left(-\frac{\sin u}{R+r\cos u}\right)d\theta_2\\
&= -\frac{\cos u(R+r\cos u)-\sin u(-r\sin u)}{(R+r\cos u)^2}du\,\theta_1\\
&\quad+\left(-\frac{\sin u}{R+r\cos u}\right)\left(-\frac{\sin u}{R+r\cos u}\right)\theta_1\wedge\theta_2\\
&= -\frac{-\cos u}{r(R+r\cos u)}\theta_1\wedge\theta_2.
\end{aligned}$$

Here we have made use of the result for $d\theta_2$ and also

$$\theta_1 = r du.$$

Solution 2.1

In the previous section we found that

$$d\omega_{12} = -\frac{\cos u}{r(R+r\cos u)}\theta_1\wedge\theta_2.$$

Comparing this with

$$d\omega_{12} = -K\theta_1\wedge\theta_2,$$

we see that

$$K = \frac{\cos u}{r(R+r\cos u)}.$$

Solution 2.2

Yes.

You may well recall that *O'Neill* has proved that the meridians and parallels of any surface of revolution are principal curves. However, since the unit normal vector field E_3 that we obtained is fairly simple, direct differentiation shows that E_1 and E_2 are principal as follows.

$$\begin{aligned}
S(E_1) &= -\nabla_{E_1}E_3\\
&= -\frac{1}{\|\mathbf{x}_u\|}\nabla_{\mathbf{x}_u}E_3 \quad \text{(definition of } E_1 \text{ and linearity)}\\
&= -\frac{1}{\|\mathbf{x}_u\|}\frac{\partial E_3}{\partial u}\\
&= -\frac{1}{\|\mathbf{x}_u\|}(\sin u\cos v, \sin u\sin v, -\cos u)\\
&= \frac{1}{\|\mathbf{x}_u\|}E_1 = \frac{1}{r}E_1.
\end{aligned}$$

Thus E_1 is an eigenvector of the shape operator and so is principal.

The calculations for $S(E_2)$ are similar.

Solution 3.1

We have

$$\begin{aligned}
\mathbf{x}_u &= (-r\sin u\cos v, -r\sin u\sin v, r\cos u),\\
\mathbf{x}_v &= (-(R+r\cos u)\sin v, (R+r\cos u)\cos v, 0),
\end{aligned}$$

and so

$$\begin{aligned}
E &= r^2,\\
\sqrt{E} &= r,\\
F &= 0, \quad \text{(orthogonal patch)}\\
G &= (R+r\cos u)^2\\
\sqrt{G} &= (R+r\cos u)\\
\frac{1}{\sqrt{EG}} &= \frac{1}{r(R+r\cos u)}.
\end{aligned}$$

Hence

$$\begin{aligned}
(\sqrt{G})_u &= -r\sin u,\\
(\sqrt{E})_v &= 0,\\
\frac{(\sqrt{G})_u}{\sqrt{E}} &= -\sin u,\\
\left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u &= -\cos u,\\
\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v &= 0.
\end{aligned}$$

Substitution now gives

$$K = \frac{\cos u}{r(R+r\cos u)}.$$

Solution 3.2

(a) Using the calculations above, we have

$$\begin{aligned}
\theta_1 &= \sqrt{E}\,du\\
&= r\,du.\\
\theta_2 &= \sqrt{G}\,dv\\
&= (R+r\cos u)dv.
\end{aligned}$$

(b) We have

$$\begin{aligned}
d\theta_1 &= dr\,du\\
&= 0,\\
d\theta_2 &= d(R+r\cos u)dv\\
&= (-r\sin u\,du + 0\,dv)dv\\
&= -r\sin u\,du\,dv.
\end{aligned}$$

(c) Substituting in

$$d\theta_1 = \omega_{12}\wedge\theta_2$$

gives

$$\begin{aligned}
0 &= (a\,du + b\,dv)\wedge((R+r\cos u)\,dv)\\
&= a(R+r\cos u)\,du\,dv.
\end{aligned}$$

It follows that $a = 0$.

Substituting into

$$d\theta_2 = -\omega_{12}\wedge\theta_1$$

gives

$$\begin{aligned}
-r\sin u\,du\,dv &= -(a\,du + b\,dv)\wedge(r\,du)\\
&= 0 - rb\,dv\,du\\
&= rb\,du\,dv.
\end{aligned}$$

Thus $b = -\sin u$ and

$$\omega_{12} = -\sin u\,dv.$$

(d) From the last answer

$$\begin{aligned}
d\omega_{12} &= d(-\sin u)\,dv\\
&= -\cos u\,du\,dv.
\end{aligned}$$

(e) We substitute into the second structural equation.

$$\begin{aligned}
d\omega_{12} &= -\cos u\,du\,dv\\
&= -K\theta_1\wedge\theta_2\\
&= -K(r\,du)\wedge((R+r\cos u)\,dv)\\
&= -Kr(R+r\cos u)\,du\,dv.
\end{aligned}$$

Hence

$$-Kr(R+r\cos u) = -\cos u$$

and so

$$K = \frac{\cos u}{r(R+r\cos u)}.$$

Solution 4.1

Since we shall require them, we calculate all the partial velocities to begin with. We suppress the parameters.

$$\begin{aligned}
\mathbf{x}_u &= (-r\sin u\cos v, -r\sin u\sin v, r\cos u),\\
\mathbf{x}_v &= (-(R+r\cos u)\sin v, (R+r\cos u)\cos v, 0),\\
\mathbf{y}_u &= (-\sin u\cos v, -\sin u\sin v, \cos u),\\
\mathbf{y}_v &= (-\cos u\sin v, \cos u\cos v, 0).
\end{aligned}$$

(a) We have

$$\begin{aligned}
\mathbf{x}_u \times \mathbf{x}_v = r(R+r\cos u)&\\
\times (-\cos u\cos v, -\cos u\sin v, -\sin u)&.
\end{aligned}$$

We are specifically asked for the outward normal. By looking at the special case $u = v = 0$, we obtain the possibilities

$$\pm(1,0,0)$$

for the normal at the point

$$(R+r, 0, 0)$$

which is on the outer 'equator' of the torus. Thus choosing the positive sign gives the outward normal which will be

$$U = (\cos u\cos v, \cos u\sin v, \sin u).$$

(b) Since

$$U(\mathbf{x}(u,v)) = \mathbf{y}(u,v),$$

F does define the Gauss map.

(c) From the calculations above, we can see that

$$\mathbf{x}_u \times \mathbf{x}_v = -r(R+r\cos u)U,$$

so

$$\begin{aligned}
\eta(\mathbf{x}_u, \mathbf{x}_v) &= \mathbf{x}_u \times \mathbf{x}_v \cdot U\\
&= -r(R+r\cos u)U \cdot U\\
&= -r(R+r\cos u).
\end{aligned}$$

From the expressions for the partial velocities of $\mathbf{y}$, and the fact that U and V have the same coordinate expressions, we have

$$\begin{aligned}
\mathbf{y}_u \times \mathbf{y}_v &= (-\cos^2 u\cos v, -\cos^2 u\sin v, -\cos u\sin u)\\
&= -\cos u\, V.
\end{aligned}$$

Hence

$$\begin{aligned}
\zeta(\mathbf{y}_u, \mathbf{y}_v) &= \mathbf{y}_u \times \mathbf{y}_v \cdot V\\
&= -\cos u\, V \cdot V\\
&= -\cos u.
\end{aligned}$$

(d) Using the definitions

$$\begin{aligned}
F^*\zeta(\mathbf{x}_u, \mathbf{x}_v) &= \zeta(F_*(\mathbf{x}_u), F_*(\mathbf{x}_v))\\
&= \zeta(\mathbf{y}_u, \mathbf{y}_v)\\
&= -\cos u.
\end{aligned}$$

From what we have calculated so far, we can say that

$$F^*\zeta(\mathbf{x}_u, \mathbf{x}_v) = \frac{\cos u}{r(R+r\cos u)}\eta(\mathbf{x}_u, \mathbf{x}_v).$$

It follows that

$$F^*\zeta = \frac{\cos u}{r(R+r\cos u)}\eta$$

and so

$$K = \frac{\cos u}{r(R+r\cos u)}.$$

Note: If you do not choose the outward normal, but do everything else consistently, you would arrive at the same result.

ISBN 0 7492 4775 4